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THE PHYSICAL EXAMINATION OF HEARING AND BINAURAL AIDS FOR THE DEAF

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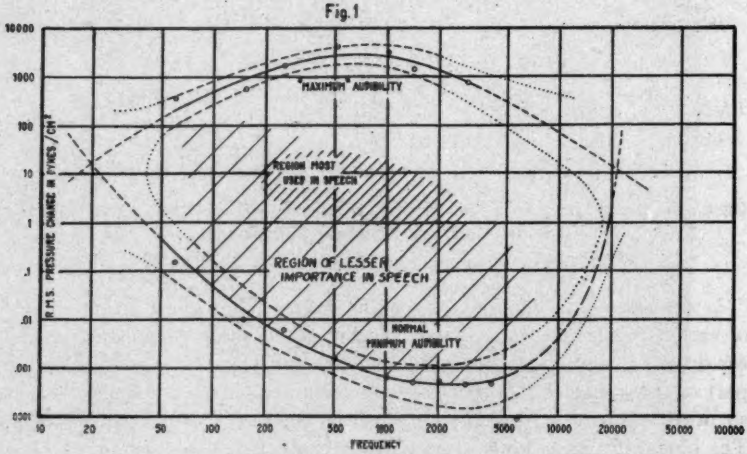
1. *Introduction.*—The function of the auditory sense is to detect sounds of various kinds and wave shapes varying over a range of pressure on the ear drum of from about .001 dyne to 1,000 dynes and over a considerable part of this range to differentiate with certainty between complex sounds so nearly alike that no existing physical apparatus can separate them. The binaural feature adds a sense of orientation with respect to a source and uniform sensitivity for sounds approaching from different directions. The abnormal auditory sense may be regarded as lacking more or less in (*a*) range of sensation (frequency and intensity); (*b*) quality of sensation in various regions of the range; (*c*) the binaural sense. Apparatus and methods have been developed by means of which the outstanding elements of these functions can be measured and to a limited extent compensated for.

In a sense this paper prefaces a more detailed discussion of the audiometrical and pathological phases of abnormal hearing which will be presented next week by Dr. E. P. Fowler and the author before meetings of the American Otological Society and the American Rhinological, Laryngological, and Otological Society.

2. *Minimum Audibility.*—Figure 1 shows a plot of the minimum audible pressure on the average of 72 normal ears taken throughout a range of frequency of from 60 to 4,000 cycles. Both the intensity and frequency scales are logarithmic. This curve has already been published (The Frequency Sensitivity of Normal Ears, by H. Fletcher and R. L. Wegel, these Proceedings, January, 1922, and in the *Physical Review*). Although all skew errors in the determination of the average curve have not been eliminated, an investigation has shown that they are so small as not to effect the utility of the curve for the purpose of measuring deafness. Among

the errors which obviously tend to raise this curve might be mentioned—lack of attention, low mentality of the observer, noise in the observing room, and abnormality of hearing. Care was taken to reduce all these errors to a minimum without actually making separate quantitative measurements of them on a rigorous statistical basis.

Fig. 1



AUDITORY SENSATION AREAS

The statistical deviation from the mean varies irregularly with frequency very likely due mostly to the external anatomical variations which cause deviations in the dynamical constants of the transmission system from sound source to the ear drum. The dotted lines following the curve of minimum audibility represent approximately the "standard deviation."

3. *Maximum Audibility.*—The curve marked "Maximum audibility" represents the pressure on the average of 48 normal ears required to produce the sensation of feeling. A sound much louder than this is painful. The measurements were taken through a range of from 60 to 3,000 cycles. The standard deviation lines are also given from which it will be seen that this curve is quite as definite as that of minimum audibility. While this point of feeling probably has no relation to the auditory sense it does serve as a practical limit to the range of auditory sensation. A few observations indicate that people with abnormal ears have a point of feeling sound which is not greatly different from that of normal ears, but this of course depends on the type of abnormality. The intensity for feeling is about equal to that required to excite the tactile nerves in the finger tips.

4. *Lower and Upper Frequency Limits of Hearing.*—The curves of mini-

mum audibility in figure 1 will be seen to have been extrapolated to the points of intersection at high and low frequencies. The feeling sensation in the middle range of frequency is first a tickling sensation and then becomes acutely painful as the loudness is increased. As the frequency is decreased the sensation of feeling becomes milder until at frequencies around 60 cycles it is sensible as a flutter, but still quite different from the sense of audition. As the frequency is still further decreased to a point where the hearing and feeling lines appear to intersect, it is difficult to distinguish between the sense of hearing and that of feeling. The low point of intersection of two normal curves of minimum audibility and feeling sense may therefore be taken arbitrarily as the lower tone limit of audibility. For frequencies lower than this it is easier to feel than to hear the air vibration. For any one individual this point cannot be determined with accuracy on account of the variation in judgment, but can be determined by extrapolation. A similar intersection of the two curves occurs at some very high frequency. The extrapolation upward of the curve of minimum audibility is consistent with some recent observations of Mr. C. E. Lane of the University of Iowa. This work will be published shortly in the *Physical Review*.

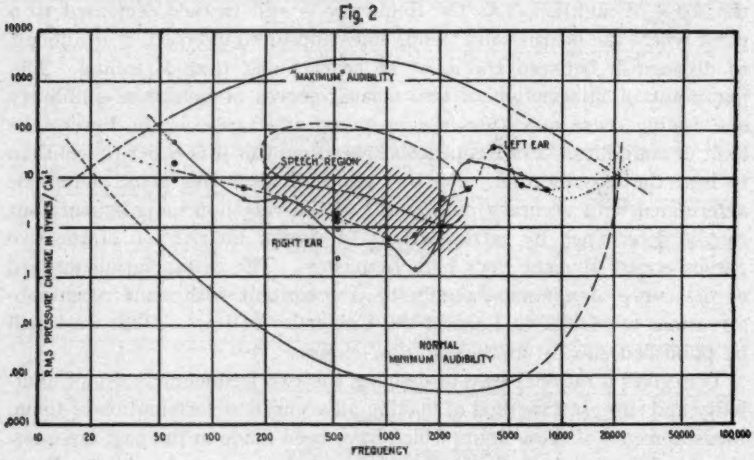
This gives a rational way of defining the two frequency limits of audibility and suggest a method of making an accurate determination of them. Measurements of these limits which have been made in the past are questionable because the intensity factor has been neglected. At the lower limit of audibility the excursions of the diaphragm and ossicles of the middle ear are probably so large that the nerves feeding these movable parts are stimulated and therefore the measurements at very low frequencies probably give data related to the pathological condition of the apparatus of the middle ear and this independently of the stimulation of the auditory nerve. This point is probably related to the tests on flexibility of the ear drum due to the application of air pressure as observed by otologists in examinations. Otologists usually associate loss of sensitivity at low frequencies with obstructive deafness if there is no loss at high frequencies.

5. *Sensation Area.*—From the combined standpoint of utility and logic the logarithmic relation between stimulus (pressure variation) and sensation can be assumed. The elliptical area between the two curves may then be taken to represent an area of sensation which is characteristic of the normal ear. Any point within this area represents a definite auditory sensation in frequency and intensity.

An abnormal ear may be regarded as having an area of sensation which is smaller than the normal area but included within it. Figure 2 is a plot of the minimum audibility of the right and left ears for a man (CHK) suffering from a catarrhal deafness. The area between this curve of mini-

mum audibility and the curve of feeling is his area of sensation. The area of sensation which is most utilized in the interpretation of speech is represented approximately by the shaded area in figures 1 and 2. (See "Analysis of the Energy Distribution in Speech," I. B. Crandall and D. MacKenzie, *Phys. Rev.*, Mar. 1922.) It will be seen that CHK retains about 50 or 60% of the normal amount of sensation. He hears and inter-

Fig. 2



SENSATION AREAS (CHK)

prets conversation with some difficulty. The CHK curves pass through the speech area. In order to make him hear well, the speech area must be raised to a higher level of intensity or loudness as indicated by the dotted curve.

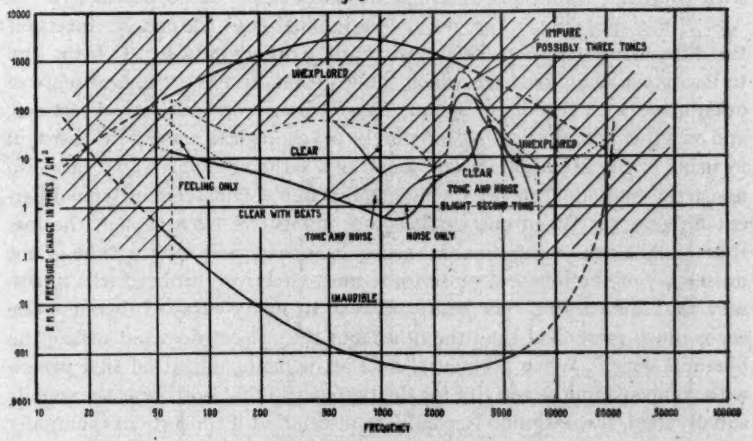
6. *Importance of Various Intensities in Speech.*—It is interesting to speculate on how CHK interprets speech. It has been shown (H. Fletcher at Franklin Institute meeting Mar. 30, 1922) that the intensity of speech may be varied over perhaps 70–80% of the range of sensation without serious loss of intelligibility to the normal ear. As the sound intensity is decreased, the intelligibility drops very suddenly to zero at minimum audibility. A similar drop is to be expected at an intensity so loud as to be painful. It is evident therefore, that the range throughout which speech is intelligible for CHK is very considerably limited as compared to normal. It is possible to design a deaf set which increases the intensity of the principal speech region to any desired place within the abnormal sensation area and so in a measure compensate for this narrowed range.

The region in figure 2 marked "Region of Lesser Importance in Conversation" corresponds to stimuli in conversation of lesser energy content such as the minor shadings and fainter consonant sounds. While it is

physically possible to produce an amplification of speech so that this region is raised into the diminished area it is impracticable to do so because of the pain which would be caused by the louder components. A diminution in sensation area can therefore be only partially compensated for. In case the area is extremely narrow a deaf set furnishing optimum volume can only serve as an aid to lip reading.

7. *Quality of Hearing.*—The sensation of a normal ear at any point in the auditory sense range (figure 1) may be described by any number of different adjectives, such as for example, "clear," "musical," "even," "sustained," "smooth," "pure," etc. Such a description may in fact be taken as a reasonable indication that the quality of sensation at the point in question is normal. Abnormal ears sometimes experience a subjective

Fig. 3



QUALITY AREAS - LEFT EAR (C.H.K)

degeneration of quality of pure stimuli which they describe as "rough," "harsh," "sharp," "buzzing," "vibrating," "hissing," etc. These sounds are independent of any tinnitus or head noises which the patient may have. Figure 3 shows various regions of the sensation area which are degenerated in the case of CHK. The shaded area was not explored. The boundaries of the degenerated regions are usually more sharply marked than those of minimum audibility and the sensations in these areas are so radically different from the sensation of a pure tone that it is with difficulty that the patient is convinced that the stimulation is the same pure tone to which he has been listening at the other intensities. The subject of these tests is a violinist and capable of better descriptions and finer distinctions than average.

Since all speech sounds may be considered as stimuli composted of various frequency components of certain intensities the sensation caused by such a sound may be represented on this plot by points, or by a line running provided the sound has a band spectrum. If the points or line falls within the sensation area the sound is audible. It is easy to see that if the points or the part of the line which represent those frequency components most essential to interpretation of the sound, fall within any of these abnormal areas, the sound is very likely to be misinterpreted. This adds a further source of loss in articulation to that already observed due to a narrowing of the sensation range.

Many practically normal ears have very small abnormal areas. They have always been found near minimum audibility and therefore have little influence on the hearing of the individual. They seem to be associated with catarrhal conditions although this cannot be stated positively.

8. *Binaural Sense.*—The normal individual has learned to interpret the differential sensations of the two ears to an advantage. It helps him to locate the direction from which sounds come, to have a sort of sense of orientation with respect to sounds approaching from different directions, and whether for physical or for purely psychological reasons to assist in focusing of the attention on one sound of a large number. Two ears also assist the individual in perceiving equally well sounds coming from different directions. When one ear becomes insensitive even though the loss is small, the use of the binaural sense disappears and after a time is not missed. For the binaural sense to be most effectively utilized it is necessary that the ears be very nearly alike. In many cases of deafness one ear is much more deaf than the other and the subject does not utilize the binaural sense. When a binaural deaf set is made and fitted to a person with compensating sensitivity for the two ears so that both hear the sounds equally loud, the sensation is usually so novel, that if the patient is actually able to experience a binaural sensation he is very much pleased. Usually, however, he has not used his binaural sense for so long a time that it takes a considerable amount of practice before he is able to have binaural experiences. It may be noted in this connection that the same experience is encountered in fitting the eyes with glasses. It is found that people with two eyes which are slightly different do not see stereoscopically but if glasses are made so as to compensate and make the eyes nearly alike it usually takes a certain time of practice before the sense of perspective can be brought back.

An audiometer similar to the one used in the measurements recorded and a binaural deaf set have been set up here for inspection.

THE PRODUCTION OF AN E.M.F. ON CLOSED CIRCUIT BY A LIGHT EFFECT ON ARGENTITE

BY H. HORTON SHELDON AND PAUL H. GEIGER

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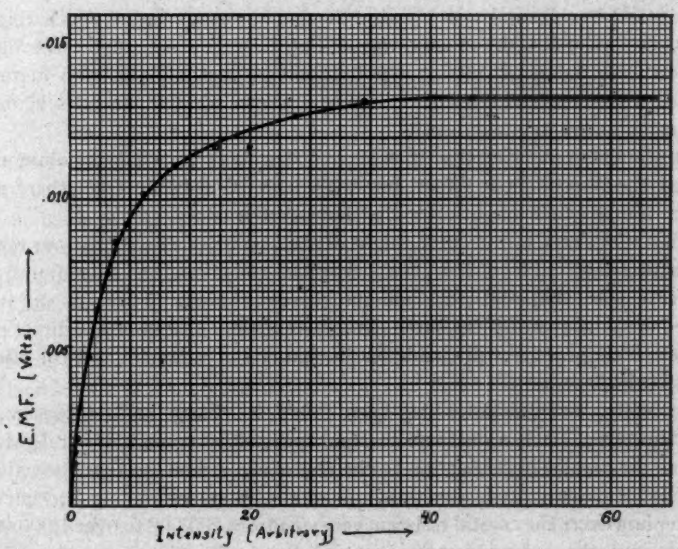
Communicated, May 12, 1922

While obtaining data on the change of resistance of Argentite (Ag_2S) with change of intensity of illumination, it was observed that under proper conditions a deflection of the galvanometer was secured on illumination when no external E.M.F. was applied to the circuit. The illuminated contact was always the negative end of the crystal.

To study this effect further, the crystal was connected directly across



the galvanometer and all factors eliminated which might in any way have an effect on the galvanometer. It was not difficult to identify the phenomenon as distinct from pressure effects, etc., but it required more careful investigation to make certain that it was not a thermoelectric effect due to the heat radiated from the lamp. The fact that the crystal turned out to be but little affected by the longer wave-lengths somewhat simplified this procedure. A heating coil radiating much more heat than the lamp by a thermopile test, was put in place of the lamp, but the result produced was extremely slight in comparison, even when the contacts were blackened by smoke from naphthalene.

The behavior of the galvanometer was also an indication of the distinction. In the case of the heater the galvanometer showed a steady creep to the maximum value, interpreted as due to the gradual heating of the exposed contact. In the case of the light the maximum was reached in approximately 13 sec., which was the quarter period of the galvanometer. A piece of plate glass an inch thick reduced the effect of light about 30%.

Having satisfied ourselves then, that the major portion of the effect was in the visible region a series of experiments were carried on with varying intensity in an attempt to find an analogy with the photo-electric effect. For this purpose a 16 c.p. bulb, such as used in an automobile head light, was employed, being chosen because of approximating closely to a point source. The intensity was varied by sliding the lamp along an optical bench, and not by any change in energy input which might vary the wave-lengths in any way. The E.M.F.'s set up by the light action were balanced out by a potentiometer method, this null method giving very consistent results. Figure 1 shows E.M.F. in volts against intensity on an arbitrary scale.

It would seem from this that it is not similar to what is commonly called the photoelectric effect, but while this is probably true, there are other things which might alter the shape of the curve.

First, the shape of the curve might depend on whether we start our readings from high or from low intensity, since resistance has been found to increase with the length of time the current flowed. This was not the case here, however, for the same curve was obtained from either direction, since a little practice will enable one to succeed in balancing out the effect very quickly.

Secondly, the resistance has been found to change with intensity of illumination. Such a decrease in resistance would cause a lower E.M.F. across the crystal in proportion to the rest of the circuit, so that the values of E.M.F. if corrected, would be raised as we go out along the curve. Since, however, the crystal resistance may be from 200,000 to over 1,000,000 ohms, and the galvanometer resistance was only 400 ohms, the effect would be small and would not materially change the shape of the curve.

It would seem then, from present data, that this is not a true photoelectric effect as the term is now used, but probably a light effect on the contact potential.

If we look again at the figure, there is to be seen a point which would suggest a bend which might indicate a secondary effect, however the accuracy of our readings do not permit of a certainty in this respect as yet. This is being investigated as well as the variation with wave-length and it is hoped that these results may be published in the near future.

The galvanometer used is the Leeds Northrup high sensitivity type

having a resistance of 400 ohms and a sensitivity of 6.02×10^{-11} amps./mm. as now used. In this work it has been critically damped. The deflections for illuminations such as used ran as high as 100 cm. The contacts used were very heavy strips of copper, about 2″ by 10″ and $^1/_8$″ thick; they were blackened by smoke from burning naphthalene except at the point of contact, and were pressed firmly against the crystal by weights placed on them. The crystals were natural and several specimens were used, obtained from the following localities: Frieberg, Saxony, Germany; Schneeberg, Saxony, Germany; Batopilas, Mexico; and Arispe Sonora, Mexico.

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*POSITIVE AND NEGATIVE PINHOLE RESONATORS**

BY CARL BARUS

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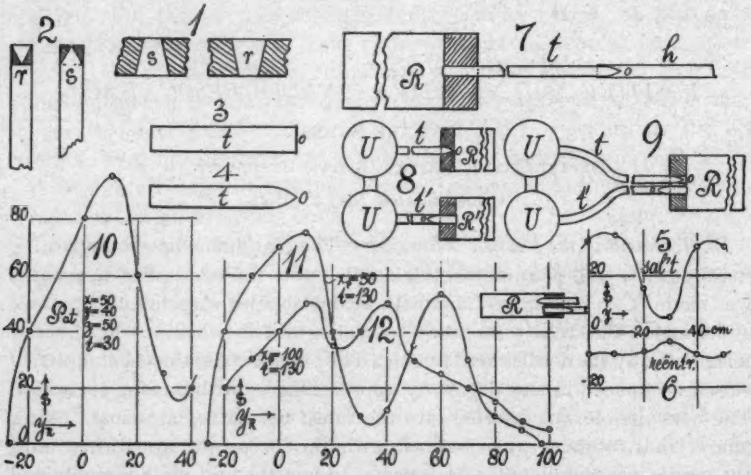
Communicated, May 11, 1922

Improvement of the Pinhole Resonator.—The plan immediately suggesting itself was the trial of an adjustable needle valve pinhole. The apparatus was made of $^1/_4$ inch brass tube, with a conical point carrying the pinhole at its end. The tube was closed by a long nut in which a waxed screw terminated by the needle was movable. The pinhole could thus be completely closed, or opened in any degree by approaching or withdrawing the screw. The new pinhole was inserted into the tuned resonator, as usual. After many trials, nothing was obtained with this promising apparatus, until the needle was completely withdrawn. In fact the best pin holes obtained for the resonator, were made from glass quill tubes drawn to a blunt closed cone. It is necessary to open and enlarge the hole by grinding; or to close the conical end by gentle fusion and regrinding when cold.

To throw further light on this intricate subject, it seemed advisable to operate with pinholes in soft sealing wax, as these could be more easily shaped. In this way much information was obtained. A thin sheet of wax was spread on thin-paper closing the end of the quill tube. When the wax was punctured by a needle from the inside of the tube, producing a conical hole as exaggerated at S in figure 1, the fringe deflections were positive and reached a maximum with the right size of hole. But when the thin wax sheet was punctured from the outside of the quill tube (suggested at r fig. 1.), the deflection obtained was to the same degree negative. In other words, in passing from the salient pinhole to the reëntrant pinhole, acoustic pressures pass to acoustic dilatations; or the pressure excess is on the reëntrant or concave side of the conical pinhole. It follows therefore that a cylindrical pinhole must be inactive, if used with a resonator.

The remarkable efficiency of glass quill tube probes, with the pin hole at a conical end, ground off, thus finds its explanation; for these are in conformity with *s*, figure 1.

To further verify the new results, holes were bored with a fluted conical reamer in small discs of brass and these were then soldered to the ends of quarter inch-brass tubes, as shown at *r* and *s* in figure 2. The cones ended in pinholes, drilled from within and without, respectively, by fine needles. The case *s* gave positive deflections, *r*, negative deflections. It was found that on enlarging *r* from within with a needle, the deflections became positive; on further enlarging the pinhole, now from without, it became



negative again. This proves, conformable with the earlier evidence, that only a very small depth of pinhole (probably of the thickness of a piece of paper) is effective and the remainder of the cone without importance.

Fine slits cut in wax behaved similarly to the pinholes above mentioned, producing pressure for a salient wedge, and the reverse. The fringe deflections, *s*, of figure 5, in which a resonator with a salient pinhole moves in *y* when tested with a reëntrant pinhole resonator, supplied the results shown in figure 6. The maxima and minima are throughout negative. Both salient and reëntrant pinholes function in the same way, with an inversion of sign, but without change of sign for the same pinhole.

The question is thus adduced, as to whether the pinhole resonator with a reversed cone, figure 7, will also produce a reversal of fringe deflections. On trial an affirmative answer was obtained at once. As a rule the negative deflections are numerically smaller than the positive deflections. It was further found that the negative deflection decreases with the length of connector tube *t*, from the bottom of the resonator *R* to the reëntrant pin-

hole O, figure 7, the pipe h beyond being the rubber connection to the U-gauge. The negative pinhole resonator runs down in sensitiveness much more rapidly (if treated in this way, pinhole remote), than if the pinhole is fixed in the bottom of the resonator, and a connector h of any length, communicates with the U-gauge.

Positive and Negative Pinhole Resonators Coöperating.—It is obvious that the positive (pinhole cone O salient) and negative (pinhole cone O' reentrant) pinhole resonators, R and R', figure 8, must add their respective pressure effects at the gauge $U\ U'$, if they are provided with independent connector tubes t and t'. A large number of such experiments all indicated that the separate deflections may be added, at least very nearly. The sensitiveness may thus be nearly doubled. It is probable moreover that two resonators will interfere less with each other than two positive or two negative resonators (i.e., two identical resonators, generally), if placed a short distance apart. The deflection in fact, falls off slowly and does not quite vanish even with a mere crevice between the mouths. If the two resonators are parallel, side by side and mouth near mouth, the conditions are even better. Hence if $R\ R'$, figure 8, are placed side by side, and the connector tubes t, t' are of pure rubber, elongated if necessary, the twin apparatus is available for exploration.

With short connectors $t\ t'$ in the interest of greater sensitiveness, I made a number of tests to find the maximum distance for which the twin resonators (fig. 8) would respond. They were able to hear an organ pipe to a distance of 6 meters, the deflection being then about a fringe.

Although a single resonator with a branched tube is ineffective, the suggestion of using the bottom of the resonator R figure 9, as a branch point for the connector tubes $t\ t'$ to the gauge $U\ U'$, seems highly promising. One and the same resonator here functions in both (positive and negative) capacities. The design succeeded with, as it seemed, but little sacrifice of sensitivity. Figure 10 shows the fringe deflections S, obtained along y, for the case of short connector tubes $t\ t'$ (30 cm. long), with the pipe (at $x = 50$, $y = 40$, $z = 50$ cm.), near the center of the table. In figure 11 the same result is extended with long connector tubes ($t = 130$ cm.). In curve 12 finally, the results refer to a more distant pipe at $x = 100$ cm., $y = 40$, $z = 50$ cm. All curves fall off quite abruptly at the two edges of the table $y = -27$ and $+93$ cm. The extreme importance of tuning was noteworthy throughout; mild notes frequently giving large deflections, sometimes out of field, whereas strong notes of the same pitch to the ear dropped the fringes back to zero. A variety of experiments were made with this ± resonator with results like the above which need not therefore be reported.

* Advance note from a Report to the Carnegie Institution of Washington, D. C.

THE EVALUATION OF QUANTUM INTEGRALS

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Communicated, May 24, 1922

Rather a long time after its publication, my attention was drawn to a paper by Mr. E. C. Kemble[1] dealing with the evaluation of the integral

$$J = 2 \int_a^b \sqrt{f(q)}\, dq, \qquad (1)$$

where a and b denote the roots of the radicand. Before preparing his own method of computation, Mr. Kemble criticizes two procedures used by other authors and rejects them as inadequate and even as erroneous. This judgment cannot remain uncontradicted for in reality these procedures are both perfectly correct and very efficient. Since the integral under consideration was first introduced into the theory of quanta by the writer of this note[2] in order to formulate his quantum conditions for conditionally periodic motions, the moral obligation of putting things right is his.

The first method to which Mr. Kemble objects is used when $f(q)$ can be expressed in the form

$$f(q) = \varphi(q) + \alpha\psi(q) \qquad (2)$$

where $\varphi(q)$ is quadratic in q, α denotes a constant and $\alpha\psi(q)$ is small compared with $\varphi(q)$. The most obvious method consists in developing $\sqrt{f(q)} = Q(\alpha)$ in powers of α

$$Q(\alpha) = Q(o) + \alpha Q'(o) + \frac{\alpha^2}{1.2} Q''(o) + \ldots. \qquad (3)$$

and integrating this series termwise. Mr. Kemble remarks that the integrals of the higher terms cannot be calculated, "because the higher derivatives of $\sqrt{f(q)}$ with respect to α become infinite at $q=a$ and $q=b$." It seems to me that the latter statement is not sufficient to draw such a conclusion; in the coefficients of series (3) there is, indeed, substituted o for α, so that they remain finite at the values $q=a$ and $q=b$, and so that the case requires a further investigation. That the roots of the function $\varphi(q)$ can also in a certain sense be regarded as limits of integration can be deduced only if the method of complex integration is used, but the latter supplies at the same time the means for overcoming the difficulty of the integrand becoming infinite at those points. This method was given by Riemann[3] just for the treatment of integrals of the type (1) and was first applied by Sommerfeld[4] to problems of the theory of quanta. It is not necessary to expand upon it for it is explained with much detail in the

books both of Riemann and of Sommerfeld.[5] Therefore the assertion of Mr. Kemble that the expansion (3) is applicable only when terms of the second and higher order in α are negligible is entirely unfounded. An expansion of this type was first introduced into the theory of quanta by the present writer in his work on the Stark effect and, as a matter of fact, the computation was carried through by him not only for the term of the first order, but also for the term of the second order[6] and proved to be as easy as can be desired.

The method followed by F. Tank in a paper referred to by Mr. Kemble is but a slight modification of this procedure. The purpose of Tank is to find an expansion for the integrand $\sqrt{f(q)}$ when it is not given in the form (2). This purpose is reached by developing $f(q)$ in a power series from the point $q=d$, which makes $f(q)$ a maximum. If $q-d=\xi$ and if H denotes the maximum value of $f(q)$, this function can be represented in the form

$$f(q) = H - \alpha\xi^2 - (\beta\xi^3 + \gamma\xi^4 + \ldots.) = H - \alpha\xi^2 - \Delta,$$

wherefrom

$$\sqrt{f(q)} = \sqrt{H-\alpha\xi^2} - \frac{1}{2}\frac{\Delta}{\sqrt{H-\alpha\xi^2}} - \frac{1}{8}\frac{\Delta^2}{(\sqrt{H-\alpha\xi^2})^3} - \ldots$$

To this series the method of complex integration can be applied termwise in exactly the same way as to expansion (3). Since the complex path of integration amounts to a real one between the roots of the radicand $H-\alpha\xi^2$, whenever this real integral is convergent. Tank was perfectly justified in taking as limits $-(H/\alpha)^{1/2}$ and $+(H/\alpha)^{1/2}$. Moreover I cannot agree with the opinion of Mr. Kemble that "the expansion is not usually convergent throughout the interval of integration." It can be shown in a general way that the problem of convergence cannot lead to any difficulties.

Summarizing, we can say that both methods in question are correct and have been successfully applied to the treatment of important problems. The new procedure proposed by Mr. Kemble may answer its purpose as a method of computation, but it surely does not equal the old ones in directness and mathematical elegance.

[1] E. C. Kemble, *Proc. Nat. Acad. Sci.*, **7**, 1921 (283).
[2] Paul S. Epstein, *Ann. Physik.*, **50**, 1916 (489).
[3] B. Riemann, *Schwere, Elektrizität, Magnetismus*, section 27, Hannover, 1880.
[4] A. Sommerfeld, *Physik. Zs.*, **17**, 1916 (500).
[5] A. Sommerfeld, *Atombau u. Spektrallinien*, pp. 476–482. Braunschweig, 1921.
[6] P. S. Epstein, *Ann. Physik.*, **51**, 1916 (184).

INVISIBLE SUN-SPOTS

By George E. Hale

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Communicated, May 26, 1922

The vortex hypothesis, proposed in 1908, assumes that a sun-spot resembles a vast tornado in which electrified particles, due to ionization in the solar atmosphere, are rapidly whirled. The invariable presence of a magnetic field, caused by the revolving charges, confirms this view,[1] which is also supported by various other results of observation with the spectrograph and spectroheliograph. Subsequently it was found that most sun-spots are associated in pairs, of opposite magnetic polarity, in which the preceding (western) spot is usually the larger of the two. 61 per cent of 970 spots examined in the years 1915–1917 were of the bipolar type, while 33 per cent were unipolar. All but 11 per cent of the unipolar spots, however, showed a tendency toward the bipolar type, indicated by trains of calcium flocculi following (less often preceding) the single member.[2] Frequently such groups oscillate between the unipolar and bipolar types, one or more small spots appearing or disappearing within the mass of calcium flocculi. This peculiarity has led to a search for invisible spots, regarded as vortices giving appreciable magnetic fields, in which the cooling due to expansion is insufficient to cause perceptible darkening of the sun's surface.

Preliminary attempts to increase the contrast by the exclusive use of ultra-violet light did not prove successful, and a second method of detecting the slight difference of radiation by a heat-measuring instrument (bolometer, thermopile, or radiometer) or a photo-electric cell has not yet been tried. A third method, however, has given very satisfactory results. This consists of a simple device for rendering weak magnetic fields visible by their Zeeman effect.

The iron line λ 6173, observed in the second order of the 75-foot spectrograph of the 150-foot tower telescope, appears as a wide triplet in the strong magnetic fields of large sun-spots. In very small spots, where the field is weak, it is merely widened, but either edge can be cut off by a Nicol prism and quarter-wave plate mounted above the slit of the spectrograph. When searching for invisible spots a superposed half-wave plate, oscillated back and forth by a small electric motor, is used to reverse the action of the quarter-wave plate when it comes before the slit. While the motor is running, a promising region on the sun (marked by faculae or calcium flocculi) is caused to pass slowly across the slit, and the λ 6173 line is carefully watched. The presence of an invisible spot is betrayed by the apparent oscillation of the corresponding part of the line, due to the al-

ternate extinction of its red and violet edges. In this way magnetic fields of an intensity of 200 gausses can be detected.

A systematic search for invisible spots was begun by Mr. Ellerman and the writer on November 19, 1921, when two were found, with negative magnetic fields of 400 and 300 gausses, respectively, in the calcium flocculi following a positive single spot, No. 1920. One of these may not have belonged to this spot group, as it was about 4° south. Both were again detected by Mr. Ellerman on the following day, with slightly weaker fields. On November 21 only one could be found, and its position was now indicated by a faint marking, which on the following day became a visible spot, with a field-strength of 300 gausses. The other invisible spot to the south was not found, but another spot, preceding it and not previously detected magnetically, had also appeared.

Several other invisible spots have since been observed. The two small spots constituting the following member of No. 1919, which disappeared to the eye on November 25, were still detected magnetically by Nicholson on November 26 and on November 27, with positive fields of 300 and 200 gausses respectively. No. 1924, recorded from visual observations as a positive unipolar spot, was found by Ellerman on December 12 to be followed by a negative magnetic field of 200 gausses; both this and another invisible spot of negative polarity, observed by him on the following day, appeared as visible spots on December 14. On January 9, 1922 Ellerman observed a positive invisible spot (300 gausses) following the eastern negative spot of a regular bipolar group, No. 1932. No spot was observed in this position either before or after January 9. A small positive spot which appeared in group No. 1938 on January 23, 1922, though not observed either visually or magnetically on the two following days, was detected by Ellerman as a positive invisible spot (400 gausses) on January 26. Following the unipolar spot No. 1942 an invisible negative spot (300 gausses) was observed on February 6 by Ellerman and on February 26 he found a positive invisible spot (500 gausses) following No. 1945 (negative) and a negative one (300 gausses) following No. 1947 (positive). No visible spots were seen in either of these positions. On March 15 Nicholson detected a negative invisible spot (300 gausses) following No. 1953 (positive) in the region where the following spots had disappeared.

In an examination of our earlier magnetic records we have found nine other cases in which a local magnetic field was observed where no spot was recorded. Most of these were probably invisible spots, but as very small visible spots at the points in question might have escaped notice, and as the magnetic observations were not followed up, they are not included here.

The systematic observation of invisible spots, especially during the periods preceding and following the visible life of those that reach ma-

turity, should assist materially in revealing the cause of spot formation. Their frequency of occurrence is probably much greater than these first results suggest, as our winter observations have been seriously limited by exceptionally cloudy weather, poor seeing, and low solar activity.

[1] Hale, On the Probable existence of Magnetic Fields in Sun-spots. *Mount Wilson Contr.*, No. 30; *Astroph. J., Chicago*, **28**, 1908 (315–343).

[2] Hale, Ellerman, Nicholson, and Joy. The Magnetic Polarity of Sun-spots. *Mount Wilson Contr.*, No. 165; *Astroph. J., Chicago*, **49**, 1919 (153–178).

COMPARISON OF THEORY WITH OBSERVATION FOR THE MINOR PLANETS 10 HYGIEA AND 175 ANDROMACHE WITH RESPECT TO PERTURBATIONS BY JUPITER

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Read before the Academy, April 24, 1922

The perturbations of the minor planets 10 Hygiea and 175 Andromache, developed by Dr. Estelle A. Glancy and Dr. Sophia H. Levy on the basis of the revision of von Zeipel's formulae and tables of minor planets which have a mean motion approximately twice that of Jupiter, (Hecuba Group), have lately been severely tested by comparison of recent observations with the computed places, for which the numerical work was performed by Dr. H. Thiele. This revision of von Zeipel's theory will soon appear as the third memoir of Volume 14 of the MEMOIRS OF THE ACADEMY. The results of these comparisons are highly encouraging and prove that the revised tables for the Hecuba Group more than meet the practical requirements of a satisfactory representation of the motion of the minor planets belonging to this group. The preliminary conclusions communicated to the Academy at the annual meeting of 1916 in my general report on the perturbations and tables of the minor planets discovered by James C. Watson are thereby fully verified.

Hygiea was discovered on April 12, 1849 by Gasparis. The most accurate of the eariler orbits computed was that by von Zech, which was based on the elements of d'Arrest and the general perturbations, developed by Hansen's method, by Jupiter, Saturn, and Mars, extending over eight oppositions. Since von Zech's death in 1864, his computations have been kept in the Recheninstitut at Berlin and were used by von Zeipel in connection with the application of his approximate perturbations for the Hecuba Group. Von Zech's computations have been continued from year to year at the Berlin Recheninstitut until 1873, up to which time they represented the motion of Hygiea with considerable accuracy. In the Berlin Jahrbuch for 1876 new elements by E. Becker are given. These

were later corrected on the basis of oppositions extending from 1868 to 1874 with consideration of the special perturbations by Jupiter and Saturn. These elements have been carried forward with perturbations until 1898 and from that time on until 1920 without change of osculation. For 1920 the elements were improved by the Berlin Recheninstitut by a slight correction of the mean motion so that the last 14 oppositions appear to be represented with an accuracy of $\pm 3'$. In the mean time, however, observations extending from 1915 to 1918 had revealed increasing discrepancies between the observations and computed places. On October 9, 1918, this discrepancy amounted to nearly $2^1/_2{}^\circ$ in right ascension and 1° in declination. These discrepancies have led to new orbits and ephemerides by Blondel, Maitre and Jehkowski, published in the Bulletin of the Observatory at Marseilles and to the latest revision of the elements by the Berlin Recheninstitut. These latter elements will no doubt represent observations at the next few oppositions with a satisfactory degree of accuracy, provided the laborious computation of the special perturbations and the correction of elements from opposition to opposition is kept up. Otherwise large discrepancies are bound to reveal themselves again in the course of a few oppositions. Even then, this piece-meal procedure does not produce a set of elements or a theory of perturbations which will represent the observations in all of the oppositions from 1849 to the present time and for at least an equally long interval in the future.

Aside from the enormous amount of computation of temporary value which is involved, it is necessary to secure frequent observations in order to secure the data for the successive improvement of the elements and perturbations. Thus, during the last ten years, more than 100 accurate photographic and visual observations of Hygiea have been published. These difficulties may now be wholly overcome by the application of von Zeipel's revised Jupiter perturbations of planets for the Hecuba Group and by the application of similar tables to other groups as originally proposed by Bohlin. As stated above, von Zeipel developed the perturbations of Hygiea from his original tables on the basis of von Zech's elements. With these perturbations he represented oppositions from 1849 to 1884. This representation showed a constantly increasing discrepancy which in 1884 had reached $23'$ in right ascension and $9'$ in declination. He therefore corrected the elements by the method of least squares and thereby succeeded in representing the observations satisfactorily until 1914. Discrepancies will hereafter rapidly increase in the opposite direction. The original discrepancies were not due so much to inaccuracy in von Zech's elements as to some inaccuracies in von Zeipel's tables for the Hecuba Group. This is clearly shown by the fact that von Zech's unchanged elements and our revised tables give as satisfactory a representation in the 37 years from 1884 to 1921 as in the preceding 35 years from 1849 to 1884,

and that several experiments by Miss Glancy, the last published in 1918 in the Astronomical Journal Number 748, to remove outstanding differences between theory and observation by correcting von Zech's elements have led to invalid results, as in the case of von Zeipel's earlier attempt in the same direction.

Outstanding differences between theory and observation are of the same order of magnitude in the two intervals. In fact in 1917 the discrepancy was less than 1′ in both right ascension and declination, although in a very few of the long series of oppositions positive or negative differences as high as 10′ exist in right ascension. These differences however, are primarily due to perturbations by Saturn which so far have not been taken into consideration. These encouraging results on Hygiea clearly show that on the basis of the revised tables for the Hecuba Group and on the basis of reliable elements determined from a very few oppositions, perturbations may be developed for all planets of this group which will yield a satisfactory theoretical representation of observations for practical purposes. It is safe to say that the abandonment of the piece-meal method of computing special perturbations and correcting elements from opposition to opposition in favor of the group theory will release in the long run a considerable part of the efforts of the astronomers who are engaged in these laborious tasks.

175 Andromache was discovered by Watson on October 1, 1877. A preliminary report on the perturbations of this planet was rendered at the annual meeting of the Academy in 1916. At that time it was pointed out that as shown by the very accurate computations by Berberich at the Recheninstitut of the special perturbations of Jupiter and Saturn, the mean motion had constantly decreased from 617.7 in 1877 to 607.9 in 1910. During the last ten years the osculating mean motion has decreased but slightly. According to Miss Levy's development of the perturbations of Andromache by the revised tables this decrease in the osculating mean motion is due in the main to three long period terms of Jupiter's perturbations having a period of approximately 187, 94, and 62 years, with an amplitude of 23.3° for the perturbation of longest period. By introducing the effect of these long period terms into the mean motion, Dr. Tittle, who has made the necessary computations, and I have approximately reproduced the nearly progressive change in the osculating mean motion as computed by Berberich. By extending this process, we may predict that the minimum value of the osculating mean motion is nearly reached and that within the next few years it will begin to increase. A case of libration is therefore not involved as the mean motion will not pass through the value which is twice that of Jupiter. The maximum value will be reached between the years 2004 and 2016, after which it will again decrease. In fact, none of the cases of planets of the Hecuba Group so far examined

in this manner indicate the existence of a libration. According to all indications, the motion of these planets is therefore entirely stable. The very extensive and accurate computations of Berberich have furnished very reliable sets of osculating elements of Andromache. With Berberich's elements osculating for 1877 and Miss Levy's development of the perturbations on the basis of the revised tables, an observation for the year 1920 has been represented with a remarkable degree of accuracy, so that in this case, also, the process of computing the special perturbations from opposition to opposition and frequent observations may be abandoned for many decades to come.

It is hoped that the results for 10 Hygiea and 175 Andromache here presented will prove a determining factor in the methods hereafter to be applied by astronomers in deriving the approximate perturbations by Jupiter of planets of the Hecuba Group and of other groups with mean motion commensurable to that of Jupiter to which Bohlin's original group theory is applicable with proper modifications and extensions. These studies should form a most fruitful field of research in theoretical astronomy.

A METHOD OF DERIVING THE DISTANCE OF THE A-TYPE STARS

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Read before the Academy, April 25, 1922

Our knowledge of the distances of individual stars of the *A*-type of spectrum is obtained almost wholly from three sources.

1. Trigonometric parallaxes.
2. Dynamical, sometimes called hypothetical, parallaxes of binary stars.
3. Parallaxes derived from group motion.

The third method probably furnishes the most accurate values for all such stars as belong to well-recognized groups. In the case of the Taurus Group, for example, the distances are known with a high degree of precision. The trigonometric parallaxes are, of course, most valuable in the case of large values for which the probable error bears a relatively small ratio to the quantities measured. The dynamical parallaxes derived from binary stars with well-determined orbital motion are of excellent quality, but are affected to some extent by the uncertainty in the value of the mass-factor for the *A*-type stars which enters directly into the computation.

A list of 100 stars with distances and absolute magnitudes obtained by these methods has formed the basis of an attempt upon our part to investigate a possible correlation between absolute magnitude and the intensities of certain spectral lines similar to that which has been used successfully in the case of stars of the later types of spectrum. The material used includes the stars for which spectra have been obtained with the Cassegrain spectrograph at Mount Wilson, and the values of the parallaxes and absolute magnitudes have been taken from the lists of trigonometric parallaxes of various observers, an unpublished investigation by Russell on dynamical parallaxes, and the memoir of Rasmuson on moving clusters. Corrections have been applied to the dynamical parallaxes to reduce them to the values corresponding to the most probable value of the mass of the A-type stars.

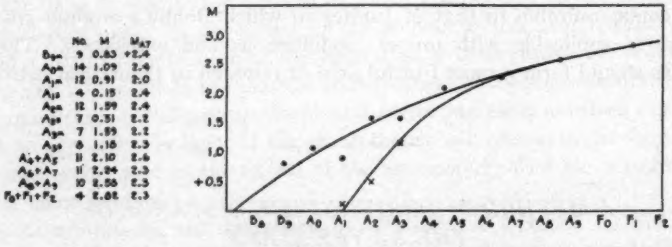

FIG. 1

Curves showing the relationship between absolute magnitude and spectral type for stars with parallaxes determined trigonometrically, dynamically and from group motions. Crosses refer to spectra with sharp lines A_1s, points to spectra with diffuse or nebulous lines as A_1n. In the present case this distinction is not carried into the more advanced sub-divisions,. MA_7 denotes the absolute magnitudes reduced to spectral type A_7 by means of these curves.

The starting point of the investigation and, as it proved, its most important feature, was an accurate determination of spectral type and the characteristics of the spectral lines. The stars were classified according to the Harvard scale as A_0, A_1, A_2, etc., and the spectra were designated *n* or *s* according as the lines are nebulous or sharp. This distinction is marked among the stars of the earlier subdivisions but becomes less certain at $A5$ or later. In addition to these two classes with sharp or nebulous lines is the small group of stars, of which α Cygni is the most prominent example, which are characterized by the Harvard observers as *c*-stars. These have exceptionally sharp and narrow lines and the enhanced lines as a rule are remarkably strong. In most of them the enhanced lines of strontium at λ4077 and λ4215 are very intense.

The following table gives the absolute magnitudes M of the stars arranged according to spectral type.

	NO.	M	MA_7		NO.	M	MA_7		NO.	M	MA_7
B9n	9	+0.83	+2.4	A2n	12	+1.59	+2.4	A4+A5	11	+2.10	2.6
A0n	14	1.09	2.4	A2s	7	0.51	2.2	A6+A7	11	2.24	2.3
A1n	12	0.91	2.0	A3n	7	1.59	2.2	A8+A9	10	2.58	2.3
A1s	4	0.15	2.4	A3s	8	1.18	2.3	$F_0+F_1+F_2$	4	2.82	2.3

An inspection of these results shows a steady increase in the values of the absolute magnitude with advancing spectral type and a marked difference in brightness between the stars of the same type according as the lines are diffuse or sharp. At type $A4$ and later no distinction is made as regard this latter characteristic except in the case of c-stars.

If these results are plotted and curves drawn through them we can from the curves read off the average absolute magnitude for each spectral type. These are as follows:

	DIFFUSE	SHARP		DIFFUSE	SHARP
B8	+0.4		A5	+2.1	+1.8
B9	0.7		A6	2.2	2.1
A0	1.0		A7	2.3	
A1	1.2	0.0	A8	2.5	
A2	1.5	0.6	A9	2.6	
A3	1.7	1.2	F0	2.7	
A4	1.9	1.5	F1	2.8	
			F2	2.9	

Finally if the absolute magnitudes of the original groups of stars are all reduced to the same spectral type $A7$ by means of these curves we obtain the results given under MA_7 in the first table. These show a remarkable degree of accordance, the total range among the groups being from 2.0 to 2.6 with an average deviation from the mean of 0.1 magnitude.

Within the groups themselves the probable error for an individual star is given below. This assumes that there is no dispersion in absolute magnitude around the mean value for a group of stars of a given spectral type with definite characteristics to their lines. That the observational results indicate this dispersion to be small is shown by the moderate size of the probable errors in spite of the fact that these are in some cases probably increased greatly by errors in the trigonometric parallaxes.

	NO.			NO.	
A9n	9	±0.48	A3n	7	±0.54
A0n	14	0.46	A3s	8	0.07
A1n	12	0.75	A4+A5	11	0.49
A1s	4	0.16	A6+A7	11	0.69
A2n	12	1.02	A8+A9	10	1.24
A2s	7	0.67	F0+F1+F2	4	0.91

The result of the comparison is to indicate that for normal A-type stars the absolute magnitude and parallax can be determined with a precision quite comparable with other methods from an accurate observation of spectral type and the characteristics of the spectral lines. For the c-stars, at least in such cases as α Cygni, this is not the case, but these stars appear to show in their spectra the phenomena connected with abnormal intrinsic brightness, and like the Cepheid variables of the F-type which they resemble noticeably there is every reason to believe they require a separate method of reduction.

Of the application of these results to individual stars it is possible to refer only very briefly but the following results may be of interest.

	TYPE	COMP. π	OBS. π
α Lyrae	A1s	0.09	0.10
ζ^1 Ursae Majoris	A2s	0.044	0.045
Taurus Group (26 stars)	A0-F	0.024	0.023
Praesepe (4 stars)	A2-A6	0.011	0.011

The close relationship of these results to certain observations by other investigators is of interest. It was found by Hertzsprung and later confirmed by Kohlschütter[2] that the stars in the Taurus cluster show approximately a linear relationship between color index or spectral type and absolute magnitude. These results deal in general with the larger spectral classes and later types and not with the subdivisions of a single type as is the case here.

In a recent investigation of the luminosity of the A-type stars, Lindblad[3] found that the intensity of the continuous spectrum in a region between λ3895 and λ3907 was less for the intrinsically fainter stars of this type than for the brighter stars. This effect he ascribes to the increased intensity of the wings of $H\zeta$ and the greater width and intensity of some of the other lines, especially those of iron and silicon, in this region. In other words the spectra of the fainter stars are diffuse as compared with those of the brighter stars, a result in agreement to that found in this investigation.

There seems to be little doubt that great intrinsic luminosity is associated with sharp and narrow lines in all spectral types and that this is due to the low density in the atmospheres of such stars. In stars of various spectral types, such as β Orionis, α Cygni, the Cepheid variables and α Orionis this effect is very marked, and the change in the character of the lines of the variable star o Ceti from maximum when they are well-defined to minimum when they are diffuse and vague is an excellent illustration of the same phenomenon.

[1] Hertzsprung, *Astron. Nachr.*, **209** (115).
[2] Kohlschütter, *Ibid.*, **211** (289).
[3] Lindblad, *Astroph. J.*, **55** (85).

THE LARGER RESULTS OF 20 YEARS OF SOLAR RADIATION OBSERVATIONS

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In Volume 4 of the *Annals of the Astrophysical Observatory* are given the results obtained from 1912 to 1920, but the discussions go back to the beginning of the investigations of solar radiation at the Smithsonian Institution in 1902. Where so many details are taken up it is a little hard to keep in mind the broad outlines, and we propose to give a general view of the work in what follows.

We measure the solar heat received by the earth; determine the limits of its variability; study the distribution of solar radiation in the spectrum, and its intensity over the solar disk; and we investigate the properties of the terrestrial atmosphere in relation to the passage of solar rays to the earth's surface and the escape of earth rays to space.

When the work was begun in 1902, great diversity of results prevailed. The black bulb in vacuo thermometer, Pouillet's pyrheliometer, Tyndall's, Violle's and Crova's actinometers, and the older form of Ångström's compensation pyrheliometer, differed through a range of at least 50 per cent in their indications. Values of the "solar constant" or intensity of solar radiation outside our atmosphere at mean solar distance ranged from 1.76 to 4.0 calories as given in the best text-books. Nothing was known as to the limits of the solar variability.

1. *Standardization and perfecting of methods and apparatus.*—While the methods we have used are Langley's they have been reduced to a finished system of every day procedure. The theory, practice, sources of error and details needful to accurate results in solar constant observing have been worked out and published in Volumes 2, 3, and 4 of the *Annals of the Astrophysical Observatory.*

Two forms of standard pyrheliometer have been devised, perfected and used to establish the scale of solar radiation measurement. Both types of instrument have hollow chamber or "black body" receivers of radiation, and in both test quantities of electrically introduced heat have been measured with close agreement.

A secondary instrument, called the silver-disk pyrheliometer, has also been perfected. About 30 standardized copies have been purchased by observers in many countries of the world. Thus the standard scale of radiation has been widely diffused. Our measurements are approximately 2 per cent higher than those of the system of Ångström with its latest corrections.

Among other instruments developed or improved for the work are the 2-mirror coelostat; the reflecting galvanometer; the vacuum bolometer for which a full mathematical theory was derived; the pyranometer, an instrument for measuring the radiation received from the daylight sky; the honeycomb pyranometer or melikeron, a "black-body" instrument for observing nocturnal radiation; the ballon pyrheliometer, 5 copies of which attached to sounding balloons, were raised to great altitudes in 1913 and 1914 for observing the solar radiation very high up in the atmosphere; the slide wire extrapolator and special plate measuring instrument, both used for the reduction of observations of the solar constant of radiation.

2. *Value of the Solar Constant.*—About 2000 observations of the intensity of solar radiation outside the atmosphere at mean solar distance, otherwise called the "solar constant," have been made according to the high and low sun method of Langley as systematized and perfected by use. These measurements have been made at Washington (sea level); Bassour, Algeria (1,160 meters); Hump Mountain, North Carolina (1,500 meters); Mt. Harqua Hala, Arizona (1,729 meters); Mt. Wilson, California (1,735 meters); Calama, Chile (2,250 meters); Montezuma, Chile (2,900 meters?); and Mt. Whitney, California (4,420 meters). The circumstances of observation have been widely varied, ranging from days of high humidity in summer at Washington to less than a millimeter of precipitable water as in the atmosphere above Mt. Whitney; at temperatures from those of the Sahara Desert in Algeria to $-25°$ C. at Hump Mountain, North Carolina; and clearness of the atmosphere, from the deep blue which prevails right up to the sun's disk on Mt. Whitney and Mt. Montezuma down to the white dusty sky which prevailed in the year 1912 at Mt. Wilson and in Algeria after the eruption of the volcano Mt. Katmai in Alaska.

These varied circumstances appear to produce very little influence on the solar constant results, and the mean value, 1.94 calories per square centimeter per minute, is closely confirmed by the observations of the balloon pyrheliometer raised to an elevation of nearly 25,000 meters where the atmospheric pressure is reduced to one twenty-fifth of its value at sea level.

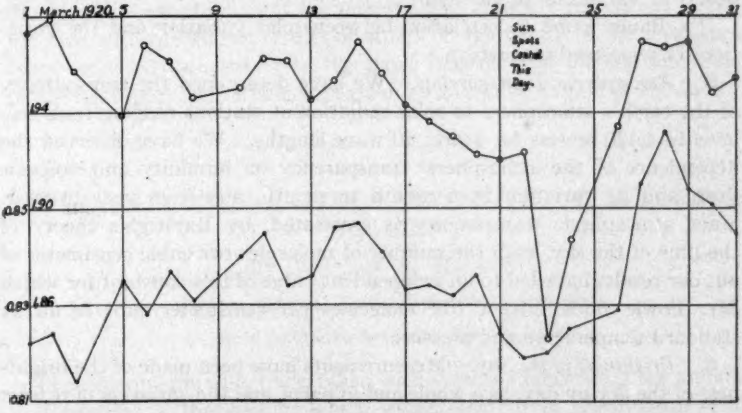
We believe this value, 1.94 calories per square centimeter per minute, may be regarded as the datum mark for this epoch with which may be compared in future centuries the solar radiation at that time prevailing in order to determine what, if any, secular changes of it have occurred.

3. *Solar Variation.*—To state the negative first, our observations have shown that the sun is not, like some of the other stars, widely variable but has at any rate not been found to vary in excess of 12 per cent in the last 20 years. In general its variations do not exceed 5 per cent in a month.

On the other hand, solar variations of from 1 to 5 per cent appear to occur at irregular intervals and with irregular magnitudes of variation;

the observed range often reaches 2 per cent within a period of ten days, and not infrequently, as, for instance at the time of the great central sun-spot group of March 22, 1920, a change of 5 per cent within 10 days is noted. The extreme range of values which has been observed by us since 1902 is approximately 12 per cent.

Long period solar variation attending the changes of solar activity, as evidenced by sun-spots, prominences, and other visible solar phenomena, appears to reach about 3 per cent for 100 Wolf sun-spot numbers. Higher solar radiation prevails at times of higher solar activity.



Solar constant and atmospheric transmission affected by great central sun-spot group, March, 1920.

Lower curve, atmospheric transparency wave-length 0.511 microns, Calama, Chile.

If *depression* of atmospheric transmission were erroneous, the result would be *increase* of solar-constant values. Apparently bombardments of earth by solar ions attending central passage of sun-spot group diminish transparency of the earth's atmosphere. This effect is frequently noted.

4. *Confirmatory Evidences of Solar Variation.*—(*a*) By studies of the distribution of radiation in the telescopic image of the sun's visible disk, fluctuations of the contrast of brightness between the center and edge of the sun have been observed for many wave-lengths. These fluctuations have complex correlation with the observed variations of the solar constant of radiation.

(*b*) In the distribution of radiation in the solar spectrum outside the atmosphere, high values appear at the shorter wave-lengths, as compared with the longer ones, when the solar radiation values are high.

(*c*) Solar radiation values themselves have been carried on at stations many thousands of miles apart and have checked one another in their indications of solar variability.

(*d*) From a rather small number of observations by Guthnick on the brightness of the planet Saturn, it appears that equal fluctuations of the brightness of the planet attend observed fluctuations of the radiation of the sun. This result, however, requires that allowance shall be made for the rotation of the sun to take into account the difference in heliographic longitude between the earth and Saturn.

(*e*) Various observers, notably Clayton, Arctowsky, Helland-Hansen, and Nansen, have published results showing the correlation between terrestrial temperatures, and other meteorological phenomena, and our observed variations of the sun.

(*f*) Bauer found a correlation between solar variation and the variations of terrestrial magnetism.

5. *Atmospheric Transparency.*—We have determined the transparency of the earth's atmosphere to solar radiation at stations ranging from sea-level to 4,420 meters for nearly 40 wave-lengths. We have observed the dependence of the atmospheric transparency on humidity and volcanic dust, and its variation from month to month, and from year to year. Since atmospheric transparency is associated, by Rayleigh's theory of the blue of the sky, with the number of molecules per cubic centimeter of air, our results have led to an independent value of this constant for which Mr. Fowle finds 2.70×10^{19} molecules per centimeter cube in air at standard temperature and pressure.

6. *Brightness of the Sky.*—Measurements have been made of the brightness of the sky by day, as a whole and in parts, and the variation of it from sea-level to 4,420 meters, as well as the effects upon it of the presence of clouds.

Mr. Aldrich took advantage of the presence of the Balloon School at Arcadia, near Mt. Wilson, to observe the reflecting power of clouds as formed there in a great level sheet some 500 meters high. From his results, it appears that if the earth were completely enshrouded in a smooth layer of cloud its reflection would be approximately 77 per cent.

7. *Long-wave Ray Investigations.*—Mr. Fowle carried on spectrobolometric investigations of the rays of the Nernst glower as affected by the humidity and carbon dioxide prevailing, in very long absorbing tubes at atmospheric pressure and temperature. These measurements covered the spectrum in the region of the earth's emission up to a wave-length of nearly 20 microns. The investigation included the distribution of the rays in the solar spectrum in this region. Many measurements have been made also on the absorption and reflection of liquid water, and many other terrestrial substances and of their degree of transparency to long-wave rays.

9. *Distribution of Radiation over the Sun's Disk.*—It was previously known that the brightness of the sun's disk falls off towards the edges, but

these measurements have been made in detail for many different wave-lengths and with the very high degree of precision necessary to observe the fluctuations of contrast of brightness from time to time. The measurements extended beyond the visible spectrum both towards the ultra-violet and towards the infra-red, and showed a striking change of distribution along the sun's diameter, depending on the difference of wave-length.

10. *Solar Theory.*—These solar radiation measurements, including those of the solar constant which have led to a value of the sun's effective temperature, those on the distribution of light over the sun's disk, and those on the transmission of the terrestrial atmosphere considered in conjunction with the known results of others on the details of the solar spectrum, have led to a theory of the sharp boundary of the sun and other things relating thereto which appears more and more to gain acceptance.

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*REPLANTATION OF ENTIRE LIMBS WITHOUT SUTURE OF VESSELS**

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Read before the Academy, April 24, 1922

The experiments were undertaken with view to determining the amount and duration of the swelling which would result from the division of all the vessels and lymphatics of a limb, our interest in the subject arising from the desire to obtain experimental support of my contention that permanent swelling of the arm following operations upon the axilla is probably invariably due to infection, an infection often so slight as to escape the observation of the surgeon. It is almost a daily experience of active surgeons to see swollen arms as result of radical operations for cancer of the breast. The swelling is frequently so great as to cause suffering and disability from the weight of the arm, and occasionally patients have consented to amputation of the entire limb for the condition which surgeons have been powerless to relieve. The ordinary swelling is universally attributed to the removal of the axillary lymphatics and veins, and some surgeons have advocated operations less thorough, believing that their results as regards the swelling of the arm have been better when the axilla was less meticulously cleaned.

Almost from the beginning of our radical operations for cancer of the breast I have been impressed with the fact that infection, however slight, was likely to be followed by more pronounced swelling of the arm and that the largest arms were noted as a rule in cases in which the inflammatory reaction and the axillary induration were greatest. We assumed, naturally,

that the excessive swelling must be due to the blocking of lymphatics and possibly veins uninjured by the knife.

About thirteen years ago a modification of the operative procedure was made, particularly in the manner of closing the wound, and thereafter, to our surprise, swollen arms became almost a thing of the past. I say "to our surprise," for the modifications in the operation were made primarily merely in the hope of securing greater freedom of motion at the shoulder joint.

Now inasmuch as the destruction of lymphatics and veins was identically the same in the modified as in the original procedure, we concluded that the most thorough excision of the axillary contents could not alone produce swelling of the arm, and that whatever the predisposing cause might be, the ultimate factor in the production of the swelling must be something which had been eliminated in the newer procedure.

It is a common observation that a moderately swollen arm may after an interval of a year or more become markedly increased in size. This augmentation is generally and quite properly attributed to a recurrence of the disease. Occasionally, but rarely, I have observed an arm that for a year or more after the operation had maintained approximately its normal size swell quite acutely. The swelling might follow closely upon constitutional symptoms—nausea, malaise, chill and fever—and be coincident with a slight or even bright blush of the skin of the shoulder, arm, and perhaps chest and forearm.

For example, in May, 1920, a patient upon whom I had performed a radical operation for mammary cancer in November, 1916, consulted me in regard to an acute swelling of the arm on the operated side. The arm had maintained its normal size prior to an "attack" in February, 1920, in the course of her convalescence from influenza. This attack as she terms it was ushered in by nausea, a chill and high fever. The arm promptly began to swell and there appeared "redness in streaks" from the shoulder to the wrist; in a few days the redness was diffuse and the swelling of the arm had become distressingly great; the hand, she said, resembled a "boxing glove," and pressure on it with the fingers produced "deep pits." In eight or ten days the redness had vanished and the swelling was decreasing. During the three months prior to this her second admission to the Johns Hopkins Hospital, there had been a less rapid reduction in the size of the affected (left) limb, which about the middle of the arm measured in circumference 9.5 cm. more than the other, and at the middle of the forearm, 3.5 cm. more. No glands were palpable above the clavicle, and there seemed to be no abnormal fullness or resistance below it. In the skin at the outer-upper edge of the grafted area there was recurrence of the carcinoma—two nodules, not ulcerated, one about the size of a filbert, the other smaller than a split pea.

The immediate cause of the swelling was undoubtedly the local infection, for during the three years and three months prior to this and subsequent to the operation there had been nothing to indicate a blocking by recurrent disease of the lymph channels.

Eight days after the second admission (June 1, 1920), the patient was operated upon by Dr. Mont Reid, our resident surgeon. The second and a part of the third portion of the axillary vein were found to be completely occluded by the new growth, which was continuous with the larger of the two cancerous nodules in the skin. The axillary vein and the recurrent growth were excised in one piece, the disease being given a wide berth. On examination of the specimen it seemed quite clear that the vein had been invaded from without, and had surely been occluded long before the attack of infection. After this operation the swelling rapidly subsided and in the course of two months the arm had almost regained its normal dimensions.

In this case, as in a number of others observed in our clinic, the occlusion of the axillary vein plus the excision of the axillary lymphatics was not followed by any swelling of the arm.

I have under observation several patients who each year following the operation have had one or two or more attacks of the kind described, each attack being followed by an increase in the size of the arm.

Only four days ago I received an announcement of the death of a patient whose story subsequent to an operation for cancer of the breast sustains the view that swelling of the arm may be due to infection even when local and general signs of inflammation are wanting. This patient was operated upon by me five years ago. About nine months after the operation she consulted me in regard to a rather acute but inconsiderable swelling of the arm on the affected side, which theretofore had not been swollen. She could not recall having had constitutional symptoms and was quite sure there had been no inflammatory redness of the arm coincident with first appearance of its enlargement. From the time of this interview until her death, four years and three months later, there had been no augmentation of the swelling. She died of carcinomatous metastases in the bones. There was no recurrence of the disease either locally or in lymphatic glands which could have affected the circulation of the arm. Hence the swelling which occurred nine months after the operation could not have been due to the blocking of lymphatics by cancer.

Operations upon the glands of the groin may be followed by elephantiasis of the scrotum. One patient in this category was operated upon by me about forty years ago. During all this period he has each year had mild local attacks of an infection resembling erysipelas and with each attack a slight, transient, increase in the elephantiasis.

Now if redness or even a faint blushing of the arm always accompanied

the advent of the swelling which occurs after an interval the discovery would undoubtedly have been made long ago of the relationship of infection to the swelling of the arm. But an infection quite negative in external manifestations other than swelling may I believe exist and suffice to produce the further blocking which in my opinion is essential to the production of the swelling.

In a word, then, I have thought it unlikely that the excision of the axillary contents, lymphatics and veins, could alone be responsible for the swelling of the arm which is observed after the radical operation for cancer of the breast in non-recurrent cases.**

Thirty-five years ago (in 1887) I successfully transplanted the hind leg of a dog from one side to the other, leaving however the main artery intact for a few days—until union between the muscles and other divided tissues had taken place. And in more recent times (1907) Carrel, with the aid of his vascular suture, made the remarkable discovery that the leg from one dog can be transplanted to another.

The operative work in the current experiments has been done by my assistants, Dr. F. L. Reichert, Dr. Mont Reid and Dr. C. Y. Bidgood. The arterial and lymphatic injections, the preparation of the specimens, the X-ray studies are all the work of Dr. Reichert.

At the outset of our experiments I had it in mind merely to determine the amount and duration of the swelling which would follow the division of all the lymphatics and veins of the thigh. At the primary operations all the soft tissues except the femoral artery and vein and the sciatic nerve were divided and immediately reunited by suture. On the second day after the replantation the femoral vein was tied in two dogs; in one of these gangrene promptly followed the ligation. Then on the fourth day after replantation the experiment of ligating the femoral vein was twice made; gangrene resulted in one of these dogs. Ligations on the 5th, 6th, 7th, 9th, and 11th days and at later periods have not been followed by gangrene; and if performed as late as the 9th day, by little or no increase in the size of the limb. Replantation and simultaneous ligation of the femoral vein was invariably followed by gangrene—6 cases.

In one case the femoral artery and vein were ligated simultaneously with the replanting—gangrene followed. Both femoral vessels may be safely ligated on the 6th day, and probably earlier.

In one instance the replantation was successful notwithstanding the fact that the femoral artery and vein had become totally obliterated by ligation of these vessels 7 months previously.

As the possibility of an anastomotic circulation by way of the bone and sciatic nerve naturally occurred to us, Dr. Reid, in one dog, divided these structures as well as the others. The replantation was successful.

To Dr. Reichert and Dr. Bidgood belongs the credit for the discovery

that replantation of an entire limb may be possible when and even when the main artery and vein have been previously ligated. The story of the successful case is as follows: November 1, 1920, Dr. Reid replanted the left hind leg of a dog. Ten days later Dr. Reichert ligated the femoral artery and vein of the opposite (right) leg for the purpose of making X-ray studies of the anastomotic circulation. On June 6, 1921, 7 months thereafter, Dr. Bidgood and Dr. Reichert amputated through the right thigh of this dog—bone undivided, having forgotten that the main vessels of this leg had been previously tied. They were surprised therefore to find that these vessels had become converted into fibrous cords and naturally feared that the replanted leg would become gangrenous. Finding later that the leg would live, we regretted that the obliterated vessels had not been severed. Hence Dr. Reichert excised a portion of each of the fibrous cords and examined them microscopically. Not only had the lumina become obliterated but the strands consisted of nothing but white fibrous tissue, the muscle and elastic tissue of the coats having disappeared. Whatever may be the explanation of this successful replantation in one act, the result is remarkable—almost beyond belief. This experiment will of course be repeated, many times if necessary, and Dr. Reichert has made the preliminary ligation of the femoral artery and vein in a number of dogs with the intention of practising the amputation and replantation at various periods thereafter.

Surgeons can best comprehend the perfection of technique required for conducting such experiments. Were it not for the fact that well nourished tissues can take care of many organisms few wounds would heal by first intention. But in the replantation experiments unusual precautions have to be taken for the reason that a devitalized limb presumably has little power to resist infection. Of the two surfaces which we are asking to unite, one is for a time quite dead if all the structures have been divided. When the artery and vein have been left for secondary ligation little imperfections in technique need not be disastrous. The impossibility of having up to the present time an absolutely perfect technique may possibly account in part for the fact that the replantations have not succeeded when all the structures have been divided except in the one instance. It is, however, astonishing, I may say hardly believable, that the complete replantation should have been successful even after preliminary ligation of the artery and vein. Having once been accomplished, an occasional repetition of the success may safely be predicted. With a technique still further perfected we might find that without preliminary ligation of the main vessels a complete replantation is occasionally possible.

After 16 days and probably earlier the new lymphatics crossing the scar are demonstrable by injection. This is perhaps the most satisfactory functional proof we have of the regeneration of lymphatics.

On the 6th day the femoral artery and vein may, as stated above, be safely ligated. Hence on the 6th day or earlier the entire circulation of the leg is carried on through the newly formed vessels—lymphatic, venous, arterial.

On the 12th day the swelling, which increases during the first week after replantation, has about disappeared.***

* The complete paper with illustrations will be published elsewhere by Dr. F. L. Reichert.

** I was greatly pleased a few weeks ago to have my attention called by Sir Almroth Wright to the following paragraph in his paper entitled: "Introduction to Vaccine Therapy:" "Lastly—for it is impossible to mention all the points, streptococcic infection would appear to be responsible for that swelling of the arm so often seen after amputation of the breast—that swelling which finds, as we are told, a perfect explanation in the 'mechanical blocking of the lymphatics.'" (Nelson Loose Leaf Medicine, 1920.)

*** Lantern pictures illustrating the author's latest method of operating for cancer of the breast, of the dogs at various periods after the replantations and of skiagrams of the injected newly formed arteries were exhibited.

DEGENERATION AND REGENERATION OF THE LATERAL-LINE ORGANS IN AMIURUS NEBULOSUS

BY MARY H. CHAMBERS

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Communicated May 31, 1922

If the nerve innervating the taste buds in the barbels of the catfish is cut, the taste buds degenerate (Olmsted, 1920b) in ten to fourteen days. Regeneration of the taste buds follows the regeneration of the nerve and is completed in about sixty days. The lateral-line organs are related anatomically to the lateral-line nerve much as the taste buds are to their nerve. Do lateral-line organs degenerate and regenerate as taste buds do?

The lateral branch of the vagus nerve innervating the lateral-line organs was severed in Amiurus. The sense organs are conspicuous structures and normally the sensory and supporting cells are easily distinguished. The first indications of degeneration appeared on the fourth day after the operation at which time the organs began to lose their distinct appearance. The sensory cells decreased in size and disappeared, first in the central part of the organ, where the nerve enters it, and later peripherally. Fragments of sensory cells may be detected for a few days longer, as degeneration in the single sense organ progresses. The supporting cells are more persistent, but by the end of two weeks the whole organ has usually disappeared.

Regeneration is more complicated than degeneration. If regeneration

of the sense organs accompanies regeneration of the nerve, one would expect to find the first regenerated sense organ in the region of the regenerated nerve. At fifty-six days there was evidence of regeneration in the sense organs and it occured very near the region of the regenerating nerve. Following the lateral-line posteriorly beyond the cut the first two sense organs in this region had not regenerated, but on the site of the third sense organ regeneration of the supporting cells had occured. Next followed a degenerated sense organ after which came a regenerated one, indistinct in outline, but with three or four sensory cells. Continuing beyond this there were no more regenerated organs.

Ninety-eight days after cutting the nerve regenerated sense organs occured in consecutive positions. The sense organs regenerate element by element as they degenerated. The supporting cells differentiate before the sensory cells do. The completely regenerated sense organs appear quite normal.

From the above data it may be concluded that degeneration of the lateral-line organs in Amiurus occurs within fourteen days after cutting the lateral nerve and that regeneration of these organs takes place after fifty-six days.

Olmsted, J. M. D. 1920a. The nerve as a formative influence in the development of taste buds. *J. Comp. Neur.*, **31**, (465–468).

Olmsted, J. M. D. 1920b. The Results of Cutting the Seventh Cranial Nerve in Amiurus nebulosus (Lesueur). *J. Exp. Zoöl.*, **31**, (369–401).

THE ISOLATION OF PYRIMIDINES FROM THE NUCLEIC ACID OF TUBERCLE BACILLUS

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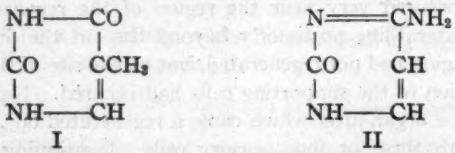
DEPARTMENT OF CHEMISTRY, YALE UNIVERSITY

Communicated May 23, 1922

This paper is a preliminary announcement of results which have been obtained in an investigation of the nucleic acid of tubercle bacillus. This research was supported in part by funds provided for the work by the National Tuberculosis Association. The raw material employed in the research was obtained through the coöperation of Dr. Esmond R. Long of Sprague Institute, Chicago, Illinois.

With the quantity of tubercle bacillus placed at our disposal it has been possible to isolate the nucleic acid in sufficient quantity for hydrolysis and to make an accurate determination of the pyrimidines contained therein. We find that this nucleic acid is characterized by the fact that it does not

contain uracil. At least we found no evidence of its presence in the product which we investigated. We have succeeded, however, in isolating in a pure condition the two pyrimidines, thymine I and cytosine II.



I II

Both pyrimidines were obtained in sufficient quantity for analysis and their constitution further established by the color reactions[1] characteristic of these two compounds, respectively. The two products isolated were compared with the pyrimidines obtained by synthesis and found to agree in every respect in chemical and physical properties. The further study of the chemistry of tubercle bacillus is in progress and the results of the investigation just finished will soon be published in The American Review of Tuberculosis and the Journal of Biological Chemistry.

[1] Wheeler and Johnson, *J. Biol. Chem.*, **3**, 1907 (183). Johnson and Baudisch, *J. Amer. Chem. Soc.*, **43**, 1921 (2670) and *Ber.*, **55**, 1921 (18).

THE RELATIVE SENSITIVITY OF THE EAR AT DIFFERENT LEVELS OF LOUDNESS

BY DONALD MACKENZIE

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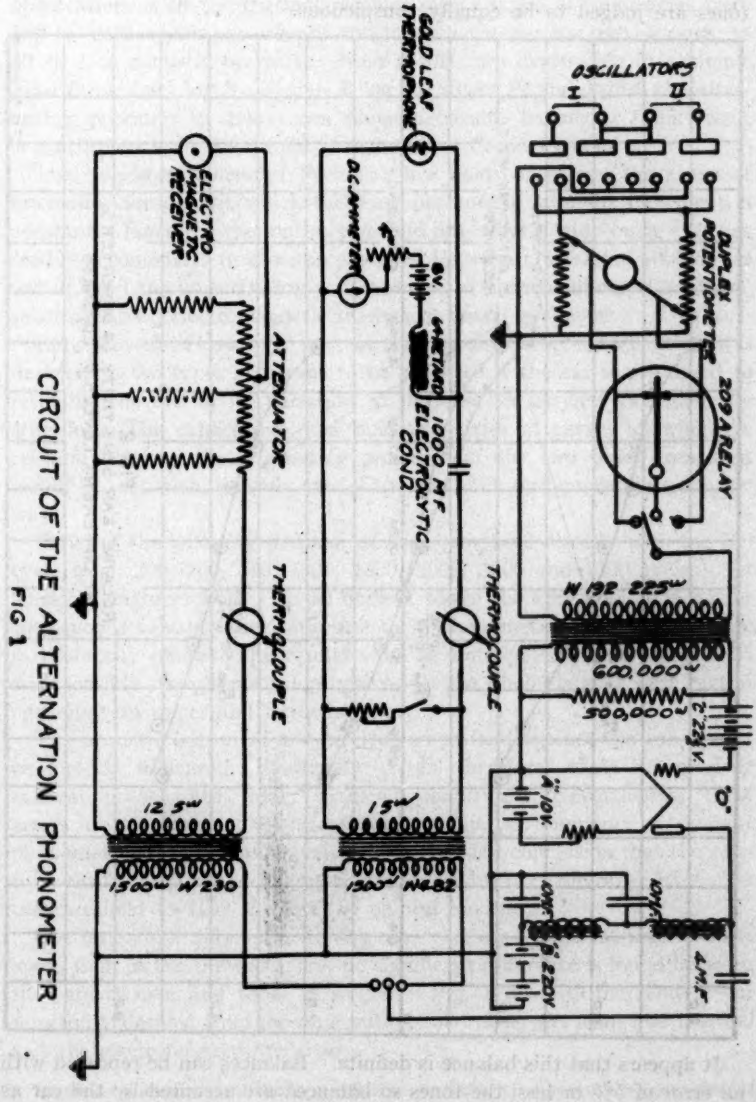
Read before the Academy, April 25, 1922

The threshold sensitivity of the ear for tones of various frequencies has been studied by many investigators. In order to determine the relative sensitivity at higher intensity levels, an alternation phonometer has been devised which permits balancing two tones of different frequency for equality of loudness.

The apparatus (fig. 1) consists of two vacuum tube oscillators, independently controlled, which through a relay commutator alternately supply current to a sound generator (either thermal or electromagnetic). The commutator is a polarized telegraph relay, driven at a suitable speed from another such relay. The commutating relay is adjusted to give a contact time of 1/25 second on each side, with a silent interval of 1/1000 second required for armature travel between contacts. This device differs in principle from the flicker photometer in that fluctuations in intensity are not caused to disappear and subjective fusion of the tones compared does not take place.* The interval (2/1000 second) through which auditory sensation persists after cessation of the physical stimulus



CIRCUIT OF THE ALTERNATION PHONOMETER

FIG. 1

is so short that we cannot use the flicker principle as applied in photometry. Accordingly, the loudness balance is found when the flickers of the two tones are judged to be equally conspicuous.

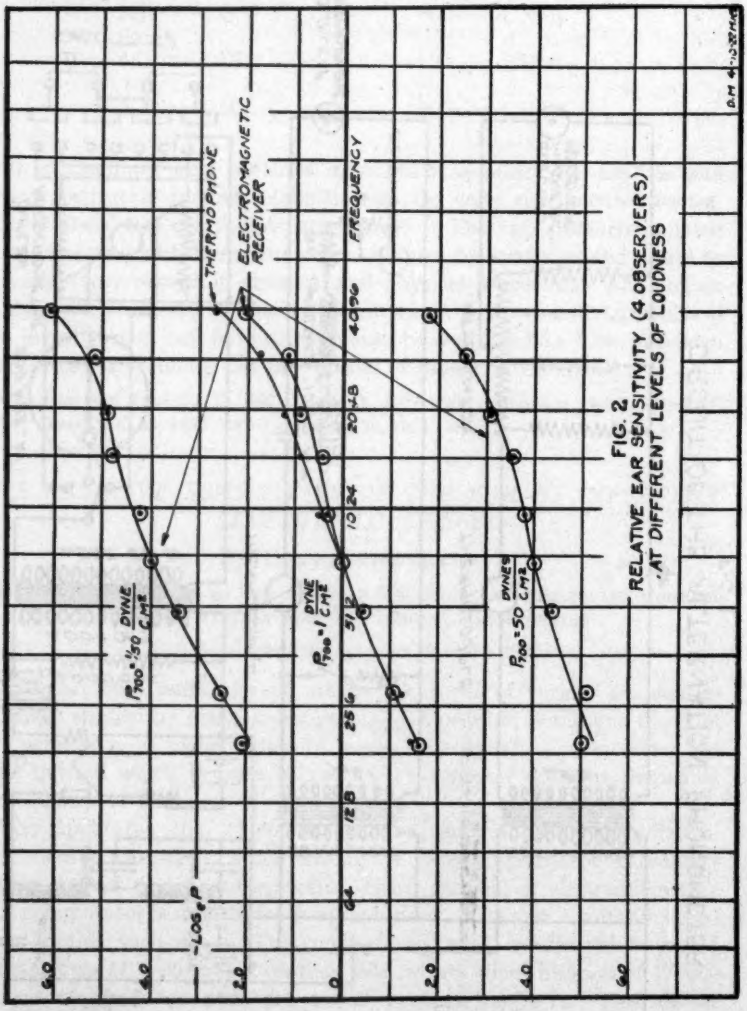


FIG. 2
RELATIVE EAR SENSITIVITY (4 OBSERVERS) AT DIFFERENT LEVELS OF LOUDNESS

It appears that this balance is definite. Balances can be repeated with an error of 5% or less; the tones so balanced are accepted by the ear as equally loud; two tones separately balanced against a third tone prove to be in balance with each other.

Twenty observers (ten men and ten women) using each one ear made observations with the thermal receiver over a range of frequencies from 100 to 4000 cycles per second, and in each comparison over a range of 10 to 1 in acoustic pressure. Their results are expressible in a simple logarithmic law: $\log P_1 = A + B \log P_2$, where P_1 and P_2 are the alternating pressures of two tones phonometrically balanced. This result is concordant with the simple formulation of Fechner's law.

The simple statement of Fechner's law holds only over the range of intensities throughout which the least perceptible stimulus increment is constant. Since the thermal receiver did not cover a wide range (the ear readily accomodates to acoustic pressures between 1/1000 and 1000 dynes per sq. cm.) the investigation was extended over wider limits of loudness, substituting an electromagnetic receiver as sound generator.

Only four observers took part in this work. Some consideration was devoted to the types of curves to be expected if the ear is presumed to resemble the eye in its variation in response to varying values of the stimulus. The expectation was that two types of curves showing the relation between the alternating pressures of the two tones compared would be met with; sensibly straight throughout, and curved in a specific way.

Tones of the following frequencies were compared directly with the 700 cycle tone: 200, 300, 500, 1000, 1500, 2000, 3000 and 4000 cycles. At levels of loudness from 1 to 10 napiers above the threshold (one napier corresponds to a pressure ratio of e to 1) phonometric comparisons were satisfactorily made. The results were as anticipated: of 32 curves, 25 were sensibly straight with slope unity over the whole range, 7 were curved oppositely in upper and lower halves.

The acoustic output of the receiver when held against the ear cannot be directly measured. Systematic errors, therefore, affect the receiver calibration obtained under artificial conditions. Nevertheless, these errors are the same at all intensities, and the predominance of straight phonometric lines of unit slope is evidence for the conclusion that the relative sensitivity of the ear is practically invariable over the range from near the threshold to near the level of painful loudness.

The individual differences among observers are smaller at intermediate levels than at the threshold, and no significant difference is found between the ears of men and those of women. Fig. 2 exhibits the relative ear sensitivity derived from the observations of 4 observers using the thermal and the electromagnetic receivers.

* A flicker phonometer similar in principle to the flicker photometer has been patented by C. M. Heck: U. S. Pat. No. 1,356,359; Oct. 19, 1920 "Method and Apparatus for for Comparing Sounds."

NORMAL COÖRDINATES FOR THE GEOMETRY OF PATHS

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Read before the Academy April 25, 1922

1. The normal coördinates introduced by Riemann have been of the greatest utility in a variety of researches in Riemann geometry and, are likely to be important in the theory of relativity. An analogous coördinate system is fundamental in what Professor Eisenhart and I have called the Geometry of Paths (Vol. 8, p. 19 of these PROCEEDINGS) i.e., in the theory of the differential equations.

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{1.1}$$

in which

$$\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha} \tag{1.2}$$

the Γ's being functions of the variables, $x^1, x^2, \ldots x^n$, and the paths being the curves which satisfy (1.1). The purpose of the present note is to define the new normal coördinates, to study a set of tensors connected with them, and to obtain a set of identities. Some of the formulae are believed to be new even for those manifolds in which the geometry of paths reduces to the Riemann geometry.

2. From the differential equations (1.1) we obtain by differentiation a sequence of differential equations.

$$\frac{d^3x^i}{ds^3} + \Gamma^i_{\alpha\beta\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{2.1}$$

$$\frac{d^4x^i}{ds^4} + \Gamma^i_{\alpha\beta\gamma\delta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds} = 0 \tag{2.2}$$

and so on, in which

$$\Gamma^i_{\alpha\beta\gamma} = \frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \Gamma^i_{j\beta}\Gamma^j_{\alpha\gamma} - \Gamma^i_{\alpha j}\Gamma^j_{\beta\gamma} \tag{2.3}$$

and in general

$$\Gamma^i_{\alpha\beta\gamma\ldots\xi\eta} = \frac{\partial\Gamma^i_{\alpha\beta\gamma\ldots\xi}}{\partial x^\eta} - \Gamma^i_{j\beta\gamma\ldots\xi}\Gamma^j_{\alpha\eta} - \Gamma^i_{\alpha j\gamma\ldots\xi}\Gamma^j_{\beta\eta} \ldots - \Gamma^i_{\alpha\beta\gamma\ldots j}\Gamma^j_{\xi\eta}. \tag{2.4}$$

3. The differential equation (1.1) has a unique set of solutions determined by the initial conditions, $x^i = p^i$ and $dx^i/ds = \xi^i$ when $s = 0$. These solutions may be written in the form:

$$x^i = p^i + \xi^i s - \frac{1}{2!}(\Gamma^i_{\alpha\beta})_P \xi^\alpha \xi^\beta s^2 - \frac{1}{3!}(\Gamma^i_{\alpha\beta\gamma})_P \xi^\alpha \xi^\beta \xi^\gamma s^3 - \ldots \quad (3.1)$$

in which the subscripts mean that the parentheses are evaluated for $x^i = p^i$, i.e., at the point P.

Let us now substitute $y^i = \xi^i s$ and solve (3.1) for y^i, obtaining

$$y^i = \psi^i(x^1, x^2 \ldots, x^n) \quad (3.2)$$

This determines a transformation of the x's into new coördinates $y^1, y^2 \ldots, y^n$. The y's are normal coördinates. They are determined uniquely by the x's, the point P, and the differential equation (1.1). They have the characteristic property that every curve

$$y^i = \xi^i s \quad (3.3)$$

is a path, i.e., a solution of the equation,

$$\frac{d^2y^i}{ds^2} + C^i_{\alpha\beta}\frac{dy^\alpha}{ds}\frac{dy^\beta}{ds} = 0 \quad (3.4)$$

into which (1.1) is transformed by the substitution (3.1). Moreover every path through P is given by (3.3).

4. Substituting (3.3) in (3.4) we obtain

$$C^i_{\alpha\beta}\xi^\alpha \xi^\beta = 0 \quad (4.1)$$

in which the functions $C^i_{\alpha\beta}$ are evaluated for the values of y such that $y^i = \xi^i s$. This is more simply written in the form

$$C^i_{\alpha\beta} y^\alpha y^\beta = 0 \quad (4.2)$$

which is an identity in y^i. Differentiating this with regard to s, we obtain

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta + 2C^i_{\alpha\gamma} y^\alpha\right)\frac{dy^\gamma}{ds} = 0. \quad (4.21)$$

Since $\frac{dy^\gamma}{ds}$ is arbitrary this gives

$$\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta + 2C^i_{\alpha\gamma} y^\alpha = 0. \quad (4.22)$$

Multiplying this by y^γ, summing, and using (4.2) we obtain

$$\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta y^\gamma = 0. \quad (4.3)$$

Differentiating once more and repeating the process just described, we obtain

$$\frac{\partial^2 C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} y^\alpha y^\beta y^\gamma y^\delta = 0, \tag{4.4}$$

and in general,

$$\frac{\partial^m C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta \ldots \partial y^\xi} y^\alpha y^\beta \ldots y^\xi = 0. \tag{4.m}$$

5. Since the directions ξ^i at the origin are entirely arbitrary it follows from (4.1) that

$$(C^i_{\alpha\beta})_0 = 0. \tag{5.2}$$

If we substitute $y^i = \xi^i s$ in (4.3), divide by s^3, and evaluate for $y^i = 0$ we have

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma}\right)_0 \xi^\alpha \xi^\beta \xi^\gamma = 0.$$

Rewriting this with α,β,γ permuted cyclically and adding, we obtain

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0 \xi^\alpha \xi^\beta \xi^\gamma = 0.$$

Since this form is symmetric and the ξ's are arbitrary,

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0 = 0. \tag{5.3}$$

By a similar argument we obtain from (4.4) the relation:

$$\left(\frac{\partial^2 C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} + \frac{\partial^2 C^i_{\alpha\gamma}}{\partial y^\beta \partial y^\delta} + \frac{\partial^2 C^i_{\alpha\delta}}{\partial y^\beta \partial y^\gamma} + \frac{\partial^2 C^i_{\beta\gamma}}{\partial y^\alpha \partial y^\delta} + \frac{\partial^2 C^i_{\beta\delta}}{\partial y^\alpha \partial y^\gamma} + \frac{\partial^2 C^i_{\gamma\delta}}{\partial y^\alpha \partial y^\beta}\right)_0 = 0. \tag{5.4}$$

By repeating this process we find that *the sum of the* $(m+2)(m+1)/2$ *derivatives of the* m^{th} *order of the functions,* C, *in which any set of* $m+2$ *integers,* $\alpha,\beta,\ldots,\xi$, $(\leqq m)$ *appear as the subscripts of the* C*'s and the superscripts of the* y*'s, is zero at the origin of normal coördinates.*

6. We now define a sequence of systems of functions of $(x^1,x^2,\ldots,x^n)$, $A^i_{\alpha\beta\ldots\xi}$, by the condition that the value of $A^i_{\alpha\beta\gamma\ldots\xi}$ at any point $(x^1,x^2,\ldots,x^n)$ of the manifold is the value of $\left(\frac{\partial^m C^i_{\alpha\beta}}{\partial y^\gamma \ldots \partial y^\xi}\right)_0$ determined in the system of normal coördinates having $(x^1,x^2,\ldots,x^n)$ as origin.

From (1.2) it follows that

$$A^i_{\alpha\beta\gamma\ldots\xi} = A^i_{\beta\alpha\gamma\ldots\xi}. \tag{6.1}$$

From the fact that all the subscripts of $A^i_{\beta\alpha\gamma\ldots\xi}$ after the first two correspond to differentiation of $C^i_{\alpha\beta}$ it follows that

$$A^i_{\alpha\beta\gamma\ldots\xi} = A^i_{\beta\alpha\delta\ldots\eta}, \tag{6.2}$$

where $\delta\ldots\eta$ stands for any permutation of the subscripts, $\gamma\ldots\xi$. From (5.3) there follows

$$A^i_{\alpha\beta\gamma} + A^i_{\beta\gamma\alpha} + A^i_{\gamma\alpha\beta} = 0. \tag{6.3}$$

This is a special case of an identity,

$$A^i_{\alpha\beta\gamma\ldots\xi} + A^i_{\alpha\gamma\beta\ldots\xi} + \ldots = 0 \tag{6.4}$$

in which there are $m(m-1)/2$ terms, each pair of the m subscripts $\alpha\beta\gamma\ldots\xi$ being the first pair in one and only one term. This identity follows directly from the last theorem of §5.

7. *The system of functions $A^i_{\alpha\beta\gamma\ldots\xi}$ with m subscripts ($m \geq 3$) is a tensor of order $m+1$.* This theorem can be inferred directly from the invariantive character of the normal coördinates. But we prefer to prove it here by showing how to express the functions A explicitly in terms of the curvature tensor and its covariant derivatives which have already been proved to be tensors in the paper by Professor Eisenhart and the writer.

8. The curvature tensor is defined by the equation,

$$B^i_{\alpha\beta\gamma} = \left(\frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \Gamma^i_{j\beta}\,\Gamma^j_{\alpha\gamma}\right) - \left(\frac{\partial\Gamma^i_{\alpha\gamma}}{\partial x^\beta} - \Gamma^i_{j\gamma}\,\Gamma^j_{\alpha\beta} = \right)\Gamma^i_{\alpha\beta\gamma} - \Gamma^i_{\alpha\gamma\beta}. \tag{8.1}$$

(This is the negative of what we denoted by $B^i_{\alpha\beta\gamma}$ in the former paper.) If it is computed in a normal coördinate system and evaluated at the origin, P, of these coördinates, it must satisfy the equations,

$$(B^i_{\alpha\beta\gamma})_P = \left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial C^i_{\alpha\gamma}}{\partial y^\beta}\right)_0 \tag{8.2}$$

These equations are equivalent to

$$(2B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta})_P = 3\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma}\right)_0 - \left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0. \tag{8.3}$$

By means of (5.3) and the definition of $A^i_{\alpha\beta\gamma}$ this leads to

$$A^i_{\alpha\beta\gamma} = \frac{1}{3}(2B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta}). \tag{8.4}$$

By means of the well-known identity,

$$B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta} + B^i_{\beta\gamma\alpha} = 0, \tag{8.5}$$

this reduces to

$$A^i_{\alpha\beta\gamma} = \frac{1}{3}(B^i_{\alpha\beta\gamma} + B^i_{\beta\alpha\gamma}). \tag{8.6}$$

The identity (8.5) is itself a direct consequence of (8.2).

9. In order to extend the formulas of §8 to our sequence of sets of functions $A^i_{\alpha\beta\gamma\ldots\xi}$ we observe that by repeated covariant differentiation of (8.1) we obtain

$$B^i_{\alpha\beta\gamma\delta\ldots\xi} = \frac{\partial^m \Gamma^i_{\alpha\beta}}{\partial x^\gamma \partial x^\delta \ldots \partial x^\xi} - \frac{\partial^m \Gamma^i_{\alpha\gamma}}{\partial x^\beta \partial x^\delta \ldots \partial x^\xi} + \ldots \qquad (9.1)$$

where the term on the left of the equality sign represents an $(m-3)$rd covariant derivative of $B^i_{\alpha\beta\gamma}$, and the three dots at the right represent terms involving derivatives of the Γ's of order less than m and covariant derivatives of $B^i_{\alpha\beta\gamma}$ of order less than $m-3$. By writing (9.1) in normal coördinates so that the Γ's in the right member become C's and then evaluating at the origin by means of the definition of the A's, we obtain

$$B^i_{\alpha\beta\gamma\delta\ldots\xi} = A^i_{\alpha\beta\gamma\delta\ldots\xi} - A^i_{\alpha\gamma\beta\delta\ldots\xi} + \ldots \qquad (9.2)$$

in which the three dots represent terms involving A's and B's with fewer than $m-1$ subscripts.

The equations (9.2) by themselves do not determine the A's uniquely in terms of the B's. When taken in connection with the equations in § 6, however, they can be solved.

10. In order to find this solution we observe that among the A's with a given set of n subscripts any one is determined by its first two subscripts, and these two subscripts are interchangeable (§ 6). We consider the following $m(m-1)/2$ permutations of the subscripts $(\beta\alpha\gamma\ldots\xi,)$ $(\gamma\beta\alpha\ldots\xi)$, $(\gamma\alpha\delta\ldots\xi)$, $(\delta\gamma\beta\ldots\xi)$, $(\delta\beta\alpha\ldots\xi)$, $(\delta\alpha\epsilon\ldots\xi)$, $(\epsilon\delta\gamma\ldots\xi)$, $(\epsilon\gamma\beta\ldots\xi)\ldots$. The A's with any two successive permutations of this set as subscripts are capable of entering in an equation of the form (9.2). There is thus determined a set of $M-1$ equations like (9.2), where $M = \frac{1}{2}m(m-1)$. Let the first of these equations be multiplied by $(M-1)$, the second by $(M-2)\ldots$ and the last by 1. On adding the resulting equations we obtain

$$\begin{aligned}(M-1)\,B^i_{\beta\alpha\gamma\ldots\xi} + (M-2)\,B^i_{\gamma\beta\alpha\ldots\xi} + (M-3)\,B^i_{\gamma\alpha\delta\ldots\xi} + \ldots \\ = MA^i_{\beta\alpha\gamma\ldots\xi} - (A^i_{\beta\alpha\gamma\ldots\xi} + A^i_{\gamma\beta\alpha\ldots\xi} + \ldots) + \ldots\end{aligned} \qquad (10.1)$$

The parenthesis of the right member is zero by (6.4). Hence (10.1) reduces to

$$A^i_{\alpha\beta\gamma\ldots\xi} = \frac{1}{M}\left\{(M-1)\,B^i_{\beta\alpha\gamma\ldots\xi} + (M-2)\,B^i_{\gamma\beta\alpha\ldots\xi} + \ldots + \ldots\right\} \qquad (10.2)$$

in which the coefficients of the terms in the parenthesis are the integers from 1 to $M-1$, the permutations of the subscripts of the B's are those indicated in the paragraph above, and the final three dots represent a

polynomial in A's and B's with fewer than $(m-1)$ subscripts. By using (10.2) as a recursion formula this last polynomial is converted into a polynomial in the B's with fewer than $(m-1)$ subscripts.

This completes the proof that each A is expressible as a polynomial in the B's and determines explicitly the coefficients of the linear expression in the B's with m subscripts which forms a part of this polynomial.

Each of the terms represented by the three dots in (9.2) is evidently a sum of products of A's and B's (such, for example, as $A^i_{\alpha\beta j}\,B^j_{\gamma\delta\epsilon}$) of such a form that it must represent a tensor if the A's and B's which enter into it represent tensors. The same remark follows directly for the terms represented by the three dots in (10.2). Hence by the use of (10.2) as a recursion formula it follows that the A's are all tensors.

11. The rule for determining the permutations of the subscripts of the $M-1$ functions B which appear linearly in (10.2) can be regarded as a rule for tracing out the points and lines of a configuration analogous to the Desargues Configuration (cf. Veblen and Young, *Projective Geometry*, Vol. I, Chap. 2). To see this it is only necessary to observe that the M permutations of the subscripts of $A^i_{\alpha\beta\gamma\ldots\xi}$ which give functions which are not identical according to (6.1) and (6.2) are in $(1-1)$ correspondence with the points of the configuration obtained by taking a plane section of a complete m-point in a projective 3-space.

12. For the case $m=3$ the formula (10.2) reduces to (8.4). For the case $m=4$ the terms represented by the three dots in (9.2) are all zero and hence they are all zero in (10.2). Hence the latter formula reduces to

$$A^i_{\alpha\beta\gamma\delta} = \frac{1}{6}\left(5B^i_{\beta\alpha\gamma\delta} + 4\,B^i_{\gamma\beta\alpha\delta} + 3\,B^i_{\gamma\alpha\delta\beta} + 2\,B^i_{\delta\gamma\beta\alpha} + B^i_{\delta\beta\alpha\gamma}\right) \qquad (11.1)$$

Another formula which follows at once from (9.1) or (11.1) is

$$A^i_{\alpha\beta\gamma\delta} = \frac{1}{24}\left[5(B^i_{\alpha\beta\gamma\delta} + B^i_{\beta\alpha\gamma\delta} + B^i_{\alpha\beta\delta\gamma} + B^i_{\beta\alpha\delta\gamma}) - (B^i_{\gamma\delta\alpha\beta} + B^i_{\delta\gamma\alpha\beta} + B^i_{\gamma\delta\beta\alpha} + B^i_{\delta\gamma\beta\alpha})\right] \qquad (11.2)$$

From (9.1) there also follows at once the identity of Bianchi,

$$B^i_{\alpha\beta\gamma\delta} + B^i_{\alpha\delta\beta\gamma} + B^i_{\alpha\gamma\delta\beta} = 0, \qquad (11.3)$$

as well as

$$B^i_{\beta\alpha\gamma\delta} + B^i_{\gamma\beta\delta\alpha} + B^i_{\delta\gamma\alpha\beta} + B^i_{\alpha\delta\beta\gamma} = 0 \qquad (11.4)$$

which is one of a sequence of important identities.

PRINCIPAL DIRECTIONS IN A RIEMANNIAN SPACE

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1. A type of principal directions for a Riemannian space of N dimensions has been defined by Ricci (*Atti R. Ist. Veneto*, **63**, p. 1233). Four distinct types are defined in the present paper, of which one is identical with that of Ricci, although defined in a manner somewhat more simple.

We shall adopt the common convention of summation with respect to any index occurring twice in a product, except where the index is a capital letter. The manifold under consideration is of N dimensions; the small Roman indices imply a range or summation from 1 to N, the small Greek indices from 1 to $N-1$. The line element being given by

$$ds^2 = g_{mn}dx_m dx_n,$$

we define in the usual manner

$$G_{mn,st} = \frac{\partial}{\partial x_t}\begin{bmatrix} ms \\ n \end{bmatrix} - \frac{\partial}{\partial x_s}\begin{bmatrix} mt \\ n \end{bmatrix} + g^{ab}\left\{\begin{bmatrix} mt \\ a \end{bmatrix}\begin{bmatrix} ns \\ b \end{bmatrix} - \begin{bmatrix} ms \\ a \end{bmatrix}\begin{bmatrix} nt \\ b \end{bmatrix}\right\} \quad (1.1)$$

$$G_{ns} = g^{mt} G_{mn,st} \quad (1.2)$$

$$G = g^{ns} G_{ns}.$$

The word "surface" will be used to denote any $(N-1)$-space immersed in the given N-space.

2. Directions defined by invariant relations may be termed *principal*. Any invariant function of direction will, in general, yield such principal directions, corresponding to stationary values of the function. The following are types of principal directions:—

Type I: Consider the family of surfaces, G = constant. Its orthogonal trajectories constitute principal directions; their equations are

$$\frac{\partial G}{\partial x_s} = \theta g_{st}\dot{x}_t \qquad (s=1,\ldots,N) \quad (2.1)$$

where the point denotes differentiation with respect to the arc. Now, for any direction,

$$\dot{G}^2 = \frac{\dfrac{\partial G}{\partial x_s}\dfrac{\partial G}{\partial x_t}dx_s\,dx_t}{g_{st}\,dx_s\,dx_t};$$

therefore directions making $\dot{G}^2$ stationary satisfy

$$\frac{\partial G}{\partial x_s}\frac{\partial G}{\partial x_t}dx_t = \phi g_{st}\,dx_t \qquad (s = 1, \ldots, N)$$

or

$$\frac{\partial G}{\partial x_r}\dot{G} = \phi g_{st}\dot{x}_t \qquad (s = 1, \ldots, N).$$

But these are of the same form as (2.1), writing $\theta\dot{G} = \phi$, and hence the type of principal direction defined above is that for which $\dot{G}^2$, and consequently $\dot{G}$, is stationary.

Type II: Consider a point (P) and a geodesic passing through it. Along this geodesic

$$\ddot{G} = \left[\frac{\partial^2 G}{\partial x_s \partial x_t} - \begin{Bmatrix} st \\ m \end{Bmatrix}\frac{\partial G}{\partial x_m}\right]\dot{x}_s\dot{x}_t.$$

This quantity is a function of direction at P, and the principal directions, corresponding to stationary values, are given by

$$\left[\frac{\partial^2 G}{\partial x_s \partial x_t} - \begin{Bmatrix} st \\ m \end{Bmatrix}\frac{\partial G}{\partial x_m}\right]dx_t = \theta g_{st}\,dx_t \qquad (s = 1, \ldots, N). \quad (2.2)$$

Type III: The expression $G_{st}\dot{x}_s\dot{x}_t$ is invariant for any given direction. Principal directions, corresponding to stationary values, are given by

$$G_{st}\,dx_t = \theta g_{st}dx_t \qquad (s = 1, \ldots, N). \quad (2.3)$$

Eisenhart (*Proc. N. A. S.*, Vol. 8, No. 2, p. 24) has shown that the principal directions of Ricci (*loc. cit.*) may be expressed in this form. These directions may also be reached from other considerations. Any direction at a point (P) defines a surface consisting of all geodesics passing through P and perpendicular to the given direction. The curvature invariant $(\bar{G})$ of this surface at P depends only on the given direction. Those directions making $\bar{G}$ stationary are principal directions: they may be proved to be identical with those considered above.

Type IV: The expression

$$g^{s_1t_1}g^{s_2t_2}g^{s_3t_3}G_{s_1s_2s_3s}\,G_{t_1t_2t_3t}\,\dot{x}_s\dot{x}_t \quad (2.4)$$

is invariant for any given direction. Principal directions, corresponding to stationary values, are given by

$$g^{s_1t_1}g^{s_2t_2}g^{s_3t_3}G_{s_1s_2s_3s}\,G_{t_1t_2t_3t}\,dx_t = \theta g_{st}\,dx_t \quad (s = 1, \ldots, N). \quad (2.5)$$

The four types defined above are not intended to be exhaustive of all types of principal directions. Type II, for example, will give principal directions if any other invariant function of position is substituted for G. Let us suppose that the expression (2.4) has the same value for all directions at any given point, but varies from point to point. The principal directions of Type IV are then indeterminate, but (2.4) is an invariant function of position and may therefore be substituted in (2.2) to yield principal directions.

3. In order to prove certain theorems concerning the principal directions defined above, we shall require particular coördinate systems. A system of coördinates will be said to be "O.T.(x_N)" if the parametric lines of x_N are the orthogonal trajectories of the family of surfaces x_N = constant. The necessary and sufficient conditions for an O.T.(x_N) system are easily seen to be

$$g_{N\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1). \tag{3.1}$$

A special type of O.T.(x_N) system is the "G.O.T.(x_N)" system for which the parametric lines of x_N are geodesics. The equations of a geodesic are

$$\ddot{x}_s + \left\{ \begin{matrix} mn \\ s \end{matrix} \right\} \dot{x}_m \dot{x}_n = 0 \qquad (s = 1, \ldots, N)$$

or

$$g_{st}\ddot{x}_t + \left[\begin{matrix} mn \\ s \end{matrix} \right] \dot{x}_m \dot{x}_n = 0 \qquad (s = 1, \ldots, N). \tag{3.2}$$

The coördinate system being O.T.(x_N), the parametric lines of x_N satisfy (3.2) if, and only if,

$$\left[\begin{matrix} NN \\ \sigma \end{matrix} \right] = 0 \qquad (\sigma = 1, \ldots, N-1)$$

and

$$g_{NN}\ddot{x}_N + \left[\begin{matrix} NN \\ N \end{matrix} \right] \dot{x}_N^2 = 0.$$

The latter equation is always satisfied for the parametric lines of x_N, by virtue of the equation

$$g_{NN}\dot{x}_N^2 = 1;$$

the former are equivalent to

$$\frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1).$$

Hence we have, as necessary and sufficient conditions for a G.O.T. (x_N) system,

$$g_{N\sigma} = 0, \quad \frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1). \tag{3.3}$$

If we are given an ∞^1 family of surfaces, we can find an O.T.(x_N) system for which the family is given by x_N = constant. If we are given a single surface and draw the congruence of geodesics normal to it, it follows from the Calculus of Variations that this is a normal congruence, and that any two of the normal surfaces give equal intercepts on all the geodesics. Taking these geodesics as parametric lines of x_N and taking x_N as the distance measured along these geodesics from the given surface, we have a

G.O.T.(x_N) system for which the given surface has the equation $x_N = 0$, and $g_{NN} = 1$ throughout the manifold (cf. Bianchi, Vol. I, p. 336).

4. A "plane" surface—the "superficie geodetiche" of Ricci (*Atti R. Acc. Lincei*, Ser. 5, Vol. 12, p. 409)—may be defined as one whose geodesics are also geodesics of the containing manifold. Let us choose an O.T. (x_N) system for which a certain plane surface has the equation $x_N = 0$. The line element of the surface is given by

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu$$

and the equations of its geodesics are, in the form of (3.2),

$$g_{\sigma\tau}\ddot{x}_\tau + \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} \dot{x}_\mu \dot{x}_\nu = 0 \qquad (4.1)$$

$$x_N = 0; \qquad (4.2)$$

for such a curve we find, for $\sigma = 1, \ldots, N-1$,

$$g_{\sigma t}\ddot{x}_t + \begin{bmatrix} mn \\ \sigma \end{bmatrix} \dot{x}_m \dot{x}_n = g_{\sigma\tau}\ddot{x}_\tau + \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix} \dot{x}_\mu \dot{x}_\nu, \quad \text{by (4.2)},$$

$$= 0, \quad \text{by (4.1)};$$

also

$$g_{Nt}\ddot{x}_t + \begin{bmatrix} mn \\ N \end{bmatrix} \dot{x}_m \dot{x}_n = \begin{bmatrix} \mu\nu \\ N \end{bmatrix} \dot{x}_\mu \dot{x}_\nu, \quad \text{by (3.1) and (4.2)}.$$

$$= -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_N}\dot{x}_\mu \dot{x}_\nu, \quad \text{by (3.1)}.$$

From the definition of a plane surface and by (3.2), this latter quantity must vanish for all arbitrary directions in the surface. Therefore we must have, at all points of the surface,

$$\frac{\partial g_{\mu\nu}}{\partial x_N} = 0 \qquad (\mu,\nu = 1, \ldots, N-1). \qquad (4.3)$$

If we are given a plane surface and choose a G.O.T.(x_N) system for which the equation of the surface is $x_N = 0$, and $g_{NN} = 1$, then, at any point of the surface, (3.3) and (4.3) hold. Applying these conditions for the reduction of (1.1) and (1.2), we find that at any point of the surface

$$G_{N\nu,\sigma t} = 0 \qquad (\nu,\sigma,t = 1, \ldots, N-1) \qquad (4.4)$$

$$G_{N\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1) \qquad (4.5)$$

5. Theorem: *The direction of Type I is contained in those of Type II if, and only if, the lines of Type I are geodesic.* In order to establish this theorem we shall employ an O.T.(x_N) system for which the surface $G =$ constant have the equations $x_N =$ constant. The principal directions of Type I are then given by

$$dx_\sigma = 0 \qquad (\sigma = 1, \ldots, N-1); \qquad (5.1)$$

those of Type II by

$$\left.\begin{aligned}\left[\frac{\partial^2 G}{\partial x_\sigma \partial x_t} - \left\{\begin{matrix}\sigma t\\ m\end{matrix}\right\}\frac{\partial G}{\partial x_m}\right] dx_t &= \theta g_{\sigma t}\, dx_t \quad (\sigma = 1, \ldots, N-1)\\ \left[\frac{\partial^2 G}{\partial x_N \partial x_t} - \left\{\begin{matrix}Nt\\ m\end{matrix}\right\}\frac{\partial G}{\partial x_m}\right] dx_t &= \theta g_{Nt}\, dx_t.\end{aligned}\right\} \tag{5.2}$$

Applying the conditions,

$$g_{N\sigma} = 0, \qquad \frac{\partial G}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1)$$

to (5.2), we obtain

$$\left.\begin{aligned} -\left\{\begin{matrix}\sigma t\\ N\end{matrix}\right\}\frac{\partial G}{\partial x_N} dx_t &= \theta g_{\sigma\tau}\, dx_\tau \quad (\sigma = 1, \ldots, N-1)\\ \frac{\partial^2 G}{\partial x_N^2} dx_N - \left\{\begin{matrix}Nt\\ N\end{matrix}\right\}\frac{\partial G}{\partial x_N} dx_t &= \theta g_{NN}\, dx_N.\end{aligned}\right\} \tag{5.3}$$

It is easily seen that (3.1) imply $g^{N\sigma} = 0$ $(\sigma = 1, \ldots, N-1)$; employing these equations we find

$$\left\{\begin{matrix}\sigma_\tau\\ N\end{matrix}\right\} dx_t = \frac{1}{2} g^{NN}\frac{\partial g_{NN}}{\partial x_\sigma} dx_N - \frac{1}{2} g^{NN}\frac{\partial g_{\sigma\tau}}{\partial x_N} dx_\tau \quad (\sigma = 1, \ldots, N-1)$$

$$\left\{\begin{matrix}Nt\\ N\end{matrix}\right\} dx_t = \frac{1}{2} g^{NN}\frac{\partial g_{NN}}{\partial x_N} dx_N + \frac{1}{2} g^{NN}\frac{\partial g_{NN}}{\partial x_\tau} dx_\tau.$$

Hence (5.3) become

$$\left.\begin{aligned} -\frac{1}{2} g^{NN}\frac{\partial G}{\partial x_N}\left(\frac{\partial g_{NN}}{\partial x_\sigma} dx_N - \frac{\partial g_{\sigma\tau}}{\partial x_N} dx_\tau\right) &= \theta g_{\sigma\tau}\, dx_\tau \quad (\sigma = 1, \ldots, N-1)\\ \left(\frac{\partial^2 G}{\partial x_N^2} - \frac{1}{2} g^{NN}\frac{\partial G}{\partial x_N}\frac{\partial g_{NN}}{\partial x_N}\right) dx_N - \frac{1}{2} g^{NN}\frac{\partial G}{\partial x_N}\frac{\partial g_{NN}}{\partial x_\tau} dx_\tau &= \theta g_{NN}\, dx_N.\end{aligned}\right\} \tag{5.4}$$

Since we hypothesize that the Type I direction is determinate,

$$\frac{\partial G}{\partial x_N} \neq 0;$$

therefore (5.4) are satisfied by (5.1) if, and only if,

$$\frac{\partial g_{NN}}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, N-1).$$

But, the coördinate system being O.T.(x_N), these conditions are necessary and sufficient that the parametric lines of x_N shall be geodesic, by (3.3). Therefore the theorem is established.

6. Theorem: *If there exists a plane surface, its normal direction is a principal direction of both Type III and Type IV.* Let us employ a G.O.T.

(x_N) system for which the equation of the plane surface is $x_N = 0$, and $g_{NN} = 1$. Since (3.3) and (4.5) hold at all points of the surface, the equations (2.3) for the directions of Type III become

$$\left.\begin{aligned} G_{\sigma\tau}\,dx_\tau &= \theta g_{\sigma\tau}\,dx_\tau \quad (\sigma = 1, \ldots, N-1) \\ G_{NN}\,dx_N &= \theta g_{NN}\,dx_N \end{aligned}\right\} \tag{6.1}$$

These equations are satisfied by $dx_\sigma = 0$ $(\sigma = 1, \ldots, N-1)$, which is the direction normal to the plane surface, and thus the theorem is established for Type III. This part of the theorem has also been proved by Ricci (*Atti R. Ist. Ven. loc. cit.*).

By (3.3) the equations (2.5) for the directions of Type IV become

$$g^{s_1t_1}\,g^{s_2t_2}\,g^{s_3t_3}\,G_{s_1s_2,s_3\sigma}\,G_{t_1t_2,t_3t}\,dx_t = \theta g_{\sigma\tau}dx_\tau \quad (\sigma = 1, \ldots, N-1) \tag{6.2}$$

$$g^{s_1t_1}\,g^{s_2t_2}\,g^{s_3t_3}\,G_{s_1s_2,s_3N}\,G_{t_1t_2,t_3t}\,dx_t = \theta g_{NN}dx_N. \tag{6.3}$$

Let us consider the surviving terms in the left hand sides of these equations, the relations effecting reductions being from (3.3),

$$g^{N\sigma} = 0 \quad (\sigma = 1, \ldots, N-1);$$

from the well known properties of the tensor-components,

$$G_{NN,st} = 0, \quad G_{st,NN} = 0 \quad (s,t = 1, \ldots, N);$$

while from (4.4) we see that any tensor-component vanishes if one and only one of its indices is N. In (6.2), if $t = N$, then either t_1 or t_2 must be N. Therefore either s_1 or s_2 must be N; therefore s_3 must be N. Hence $t_3 = N$, and the second tensor vanishes. Therefore there are no surviving terms in (6.2) for which $t = N$, and (6.2) may be written

$$A_{\sigma\tau}\,dx_\tau = \theta g_{\sigma\tau}\,dx_\tau \quad (\sigma = 1, \ldots, N-1). \tag{6.4}$$

In (6.3) either s_1 or s_2 must be N. Therefore either t_1 or t_2 must be N; therefore either t_3 or t must be N. But if $t_3 = N$, then $s_3 = N$ and the term vanishes. Therefore $t_3 \neq N$, and the only surviving term is that for which $t = N$. Thus (6.3) may be written

$$B dx_N = \theta g_{NN}\,dx_N. \tag{6.5}$$

But (6.4) and (6.5) are satisfied simultaneously by $dx_\sigma = 0$ $(\sigma = 1, \ldots, N-1)$ and thus the theorem is proved.

In the case $N = 2$, the directions of Types III and IV become indeterminate, since through any point we can draw a "plane surface" (in this case a geodesic curve) so that its normal may have any arbitrarily assigned direction.

PRINCIPAL DIRECTIONS IN THE EINSTEIN SOLAR FIELD

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In the Newtonian manifold of space-time there is at any point one principal direction, namely that for which the three space coördinates are stationary. In the hyperbolic space-time of the Special Relativity Theory, there are no principal directions, the only directions intrinsically definable being those forming the cone

$$-dx^2 - dy^2 - dz^2 + c^2dt^2 = 0.$$

In the previous paper (*Proc. N. A. S.*, p. 198) several types of principal directions have been defined for a Riemannian N-space, but the field equations

$$G_{mn} = 0$$

render indeterminate the principal directions of Types I, II and III as there defined. Eisenhart (*Proc. N. A. S.*, Vol. 8, No. 2, p. 24) has shown that those of Type III are indeterminate for all three forms of Einstein space free from matter. However, it would appear that those of Type IV might exist in the Solar Field; they correspond to stationary values of the invariant function of direction

$$\theta = g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2s_3s} G_{t_1t_2t_3t} \frac{dx_s}{ds} \frac{dx_t}{ds}.$$

But, as will be seen, the value of θ at any point proves to be independent of direction and therefore the principal directions of Type IV are indeterminate. However, since θ is an invariant function of position varying from point to point, there will exist principal directions corresponding to stationary values of $d^2\theta/ds^2$ for geodesics drawn in all possible directions. These principal directions are given by

$$\left[\frac{\partial^2\theta}{\partial x_s \partial x_t} - \begin{Bmatrix} st \\ m \end{Bmatrix} \frac{\partial\theta}{\partial x_m}\right] dx_t = \phi g_{st}\, dx_t \quad (s = 1, 2, 3, 4) \qquad (1)$$

and are a generalization of Type II.

The manifold under consideration is of four dimensions, and, in accordance with the conventions employed in the foregoing paper small Roman indices imply a range or summation from 1 to 4, small Greek indices from 1 to 3. The line element is given by

$$ds^2 = g_{mn}\, dx_m\, dx_n$$

where

$$g_{11} = -(1-k/x_1)^{-1}, \quad g_{22} = -x_1^2, \quad g_{33} = -x_1^2 \sin^2 x_2,$$
$$g_{44} = 1-k/x_1, \quad g_{mn} = 0 \quad (m \neq n).$$

Hence we have

$$g^{mn} = \begin{cases} 1/g_{mn} & (m = n) \\ 0 & (m \neq n). \end{cases}$$

Observing that

$$\frac{\partial g_{st}}{\partial x_3} = \frac{\partial g_{st}}{\partial x_4} = 0,$$

we find that any three index symbol, $\begin{bmatrix} mn \\ s \end{bmatrix}$, is zero if just one of the indices is either 3 or 4. Turning to the general expression

$$G_{mn,st} = \frac{\partial}{\partial x_t}\begin{bmatrix} ms \\ n \end{bmatrix} - \frac{\partial}{\partial x_s}\begin{bmatrix} mt \\ n \end{bmatrix} + g^{ab}\left\{\begin{bmatrix} mt \\ a \end{bmatrix}\begin{bmatrix} ns \\ b \end{bmatrix} - \begin{bmatrix} ms \\ a \end{bmatrix}\begin{bmatrix} nt \\ b \end{bmatrix}\right\} \quad (2)$$

we find

$$G_{4\nu,\sigma\tau} = g^{ab}\left\{\begin{bmatrix} 4\tau \\ a \end{bmatrix}\begin{bmatrix} \nu\sigma \\ b \end{bmatrix} - \begin{bmatrix} 4\sigma \\ a \end{bmatrix}\begin{bmatrix} \nu\tau \\ b \end{bmatrix}\right\}.$$

Now $\begin{bmatrix} 4\tau \\ a \end{bmatrix}$ vanishes unless $a = 4$, $\begin{bmatrix} \nu\sigma \\ b \end{bmatrix}$ vanishes if $b = 4$, while g^{ab} vanishes unless $a = b$. Therefore

$$g^{ab}\begin{bmatrix} 4\tau \\ a \end{bmatrix}\begin{bmatrix} \nu\sigma \\ b \end{bmatrix} = 0;$$

applying similar reasoning to the second part of the expression, we find

$$G_{4\nu,\sigma\tau} = 0.$$

Thus any tensor-component with just one index equal to 4 vanishes. Similarly any tensor-component with just one index equal to 3 vanishes.

From (2) we find

$$G_{4\nu,\sigma 4} = \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_\nu \partial x_\sigma} - \frac{1}{2}g^{11}\begin{bmatrix} \nu\sigma \\ 1 \end{bmatrix}\frac{\partial g_{44}}{\partial x_1} - \frac{1}{4}g^{44}\frac{\partial g_{44}}{\partial x_\nu}\frac{\partial g_{44}}{\partial x_\sigma}, \quad (3)$$

and hence

$$G_{4\nu,\sigma 4} = 0 \quad (\nu \neq \sigma).$$

The surviving members of the class given in (3) are

$$\left.\begin{aligned} G_{41,14} &= \frac{1}{2}\frac{\partial^2 g_{44}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{11}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} - \frac{1}{4}g^{44}\left(\frac{\partial g_{44}}{\partial x_1}\right)^2 = -\frac{k}{x_1^3} \\ G_{42,24} &= \frac{1}{4}g^{11}\frac{\partial g_{22}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} = \frac{1}{2}\frac{k}{x_1}\left(1 - \frac{k}{x_1}\right) \\ G_{43,34} &= \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{44}}{\partial x_1} = \frac{1}{2}\frac{k}{x_1}\sin^2 x_2\left(1 - \frac{k}{x_1}\right) \end{aligned}\right\} \quad (4)$$

From (2) we find

$$G_{\mu\nu,\sigma\tau} = \frac{\partial}{\partial x_\tau}\begin{bmatrix}\mu\sigma\\ \nu\end{bmatrix} - \frac{\partial}{\partial x_\sigma}\begin{bmatrix}\mu\tau\\ \nu\end{bmatrix} + g^{\alpha\beta}\left\{\begin{bmatrix}\mu\tau\\ \alpha\end{bmatrix}\begin{bmatrix}\nu\sigma\\ \beta\end{bmatrix} - \begin{bmatrix}\mu\sigma\\ \alpha\end{bmatrix}\begin{bmatrix}\nu\tau\\ \beta\end{bmatrix}\right\}. \quad (5)$$

Since the presence of just one 3 among the indices makes the tensor-component vanish, the surviving independent members of this class are

$$G_{31,13}, \quad G_{31,23}, \quad G_{32,23}, \quad G_{21,12};$$

we calculate them from (5):—

$$\left.\begin{aligned} G_{31,13} &= \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{11}}{\partial x_1} - \frac{1}{4}g^{33}\left(\frac{\partial g_{33}}{\partial x_1}\right)^2 = -\frac{1}{2}\sin^2 x_2\,\frac{k/x_1}{1-k/x_1} \quad (6)\\ G_{31,23} &= \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_1 \partial x_2} - \frac{1}{4}g^{22}\frac{\partial g_{33}}{\partial x_2}\frac{\partial g_{22}}{\partial x_1} - \frac{1}{4}g^{33}\frac{\partial g_{33}}{\partial x_2}\frac{\partial g_{33}}{\partial x_1} = 0\\ G_{32,23} &= \frac{1}{2}\frac{\partial^2 g_{33}}{\partial x_2^2} + \frac{1}{4}g^{11}\frac{\partial g_{33}}{\partial x_1}\frac{\partial g_{22}}{\partial x_1} - \frac{1}{4}g^{33}\left(\frac{\partial g_{33}}{\partial x_2}\right)^2 = kx_1\sin^2 x_2\\ G_{21,12} &= \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x_1^2} - \frac{1}{4}g^{11}\frac{\partial g_{22}}{\partial x_1}\frac{\partial g_{11}}{\partial x_1} - \frac{1}{4}g^{22}\left(\frac{\partial g_{22}}{\partial x_1}\right)^2 = -\frac{1}{2}\frac{k/x_1}{1-k/x_1} \end{aligned}\right\} \quad (7)$$

The complete list of surviving components, derivable from (4), (6) and (7), is as follows:—

of type $G_{s_1 s_2, s_3 1}$ $\begin{Bmatrix} G_{21,21} & G_{12,21}\\ G_{31,31} & G_{13,31}\\ G_{41,41} & G_{14,41}\end{Bmatrix}$; of type $G_{s_1 s_2, s_2 2}$ $\begin{Bmatrix} G_{12,12} & G_{21,12}\\ G_{32,32} & G_{23,32}\\ G_{42,42} & G_{24,42}\end{Bmatrix}$;

of type $G_{s_1 s_2, s_3 3}$ $\begin{Bmatrix} G_{13,13} & G_{31,13}\\ G_{23,23} & G_{32,23}\\ G_{43,43} & G_{34,43}\end{Bmatrix}$; of type $G_{s_1 s_2, s_2 4}$ $\begin{Bmatrix} G_{14,14} & G_{41,14}\\ G_{24,24} & G_{42,24}\\ G_{34,34} & G_{43,34}\end{Bmatrix}$.

The equations of the principal directions of Type IV are

$$\theta g_{st}\,dx_t = g^{s_1 t_1} g^{s_2 t_2} g^{s_3 t_3} G_{s_1 s_2, s_3 s}\, G_{t_1 t_2, t_3 t}\, dx_t \quad (s = 1, 2, 3, 4); \quad (8)$$

these become

for $s = 1$, $\theta g_{11} dx_1 = 2g^{11}\left[(g^{22}G_{21,12})^2 + (g^{33}G_{31,13})^2 + (g^{44}G_{41,14})^2\right] dx_1$;

for $s = 2$, $\theta g_{22} dx_2 = 2g^{22}\left[(g^{11}G_{12,21})^2 + (g^{33}G_{32,23})^2 + (g^{44}G_{42,24})^2\right] dx_2$;

for $s = 3$, $\theta g_{33} dx_3 = 2g^{33}\left[(g^{11}G_{13,31})^2 + (g^{22}G_{23,32})^2 + (g^{44}G_{43,34})^2\right] dx_3$;

for $s = 4$, $\theta g_{44} dx_4 = 2g^{44}\left[(g^{11}G_{14,41})^2 + (g^{22}G_{24,42})^2 + (g^{33}G_{34,43})^2\right] dx_4$.

On substitution we obtain

$$\theta dx_1 = 3\frac{k^2}{x_1^6}\,dx_1, \qquad \theta dx_2 = 3\frac{k^2}{x_1^6}\,dx_2,$$

$$\theta dx_3 = 3\frac{k^2}{x_1^6}\,dx_3, \qquad \theta dx_4 = 3\frac{k^2}{x_1^6}\,dx_4.$$

Thus the principal directions of Type IV are indeterminate, and (8) define an invariant function of position

$$\theta = g^{s_1 t_1} g^{s_2 t_2} g^{s_3 t_3} G_{s_1 s_2 s_3} G_{t_1 t_2 t_3} \frac{dx_s}{ds} \frac{dx_t}{ds} = 3 \frac{k^2}{x_1^6}.$$

Substituting this value for θ in (1), we obtain, after reduction, the following equations for principal directions:—

$$\phi dx_1 = 9 \frac{k^2}{x_1^8} \left(15 \frac{k}{x_1} - 14\right) dx_1, \qquad \phi dx_2 = 18 \frac{k^2}{x_1^8} \left(1 - \frac{k}{x_1}\right) dx_2,$$

$$\phi dx_3 = 18 \frac{k^2}{x_1^8} \left(1 - \frac{k}{x_1}\right) dx_3, \qquad \phi dx_4 = 9 \frac{k^2}{x_1^9} dx_4.$$

These equations determine the following directions:—

(i) the parametric lines of x_1, $(dx_2 = dx_3 = dx_4 = 0)$;
(ii) any direction making $dx_1 = dx_4 = 0$;
(iii) the parametric lines of x_4, $(dx_1 = dx_2 = dx_3 = 0)$.

It might be said that these principal directions illustrate both the radial and the stationary characters of the field.

FIELDS OF PARALLEL VECTORS IN THE GEOMETRY OF PATHS

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1. In a former-paper (these PROCEEDINGS, Feb. 1922) Professor Veblen and the writer considered the geometry of a general space from the point of view of the paths in such a space—the paths being a generalization of straight lines in euclidean space. From this point of view it is natural to think of the tangents to a path as being parallel to one another. In this way our ideas may be coördinated with those of Weyl and Eddington who have considered parallelism to be fundamental rather than the paths which we so consider. It is the purpose of this note to determine the geometries which possess one or more fields of parallel vectors, which accordingly define a significant direction, or directions, at each point of the space.

2. The equations of the paths are taken in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \qquad (2.1)$$

where x^i $(i = 1, \ldots n)$ are the coördinates of a point of a path expressed as functions of a parameter s; $\Gamma^i_{\alpha\beta}$ are functions of the x's such that $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$.

The components dx^i/ds of the vector tangent to a path being contravariant we put $dx^i/ds = A^i$. In the former paper we observed that the theory of covariant differentiation can be generalized to the geometry of paths by replacing the Christoffel symbols $\left\{ {jk \atop i} \right\}$ by Γ^i_{jk}. Thus the quantities

$$A^i_j = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha \tag{2.2}$$

are the covariant derivatives of A^i; they are the components of a mixed tensor of the second order. Thus $A^i_j\, dx^j$ expresses in invariant form the first variation of the components of A^i as the x's vary. Hence if we write (2.1) in the form

$$A^i_j\, dx^j/ds = 0, \tag{2.3}$$

we see that the first variation of the components of the tangent vector to a path is equal to zero. In this sense we speak of the tangents to a path as parallel.

Suppose now that A^i are the components of any contravariant vector whatever, and consider the vectors at points of any curve C not necessarily a path. The components A^i and coördinates x^i along C are expressible in terms of a parameter s, and dx^i/ds are the components of the tangent to C. If these functions are such that equations (2.3) are satisfied, we say that the vectors A^i are *parallel to one another with respect to the curve*. In particular the tangents to a path are parallel with respect to it. Some time ago Professor Veblen, in discussing the covariant derivative of a tensor, pointed out that it should be interpreted as the system of turning components of the given tensor with respect to the given direction. In this sense (2.3) expresses the fact that the turning components of the vector along the curve are zero.

In order that our definition may be such that if A^i are the components of parallel vectors with respect to a curve so also are φA^i, where φ is a scalar, we say that the vectors of components A^i are parallel with respect to a curve whose tangents have the components dx^j/ds, provided that a scalar function φ exists such that

$$\left(\frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha - \frac{\partial \log \varphi}{\partial x^j} A^i \right) \frac{dx^j}{ds} = 0. \tag{2.4}$$

3. Equation (2.4) is satisfied independently of the curve, if

$$\frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha - \frac{\partial \log \varphi}{\partial x^j} A^i = 0, \tag{3.1}$$

that is

$$A^i_j - \frac{\partial \log \varphi}{\partial x^j} A^i = 0. \tag{3.2}$$

Consequently whenever there exists a field of vectors satisfying (3.1), all the vectors are parallel to one another for any curve, and thus there is a significant direction at each point of the space.

For the second covariant derivatives of any contravariant vector A^i we have the identity

$$A^i_{jk} - A^i_{kj} = - A^\alpha B^i_{\alpha jk}, \tag{3.3}$$

where

$$B^i_{\alpha jk} = \frac{\partial \Gamma^i_{\alpha k}}{\partial x^j} - \frac{\partial \Gamma^i_{\alpha j}}{\partial x^k} + \Gamma^\beta_{\alpha k}\Gamma^i_{\beta j} - \Gamma^\beta_{\alpha j}\Gamma^i_{\beta k}, \tag{3.4}$$

that is the B's are the components of the curvature tensor, as defined in the former paper. From (3.3) it follows that the conditions of integrability of (3.1) are

$$A^\alpha B^i_{\alpha jk} = 0. \tag{3.5}$$

From this equation it follows that a necessary and sufficient condition that (3.1) be completely integrable, that is that there exists a field of vectors parallel to any given vector is $B^i_{\alpha jk} = 0$. From the results of the former paper it follows that in this case the space is euclidean.

If the space is not euclidean, a necessary condition that the A's given by (3.5) shall satisfy (3.2) is

$$A^\alpha B^i_{\alpha jkl} = 0, \tag{3.6}$$

where $B^i_{\alpha jkl}$ is the covariant derivative of $B^i_{\alpha jk}$.

Suppose now that the rank of the matrix of equations (3.5) is such that these equations admit a set of solutions A^i determined to within a scalar factor, and that these functions satisfy (3.6). Differentiating (3.5) covariantly with respect to x^l and taking account of (3.6), we have that the functions A^α_l must satisfy (3.5). In consequence of the above assumptions, it follows that

$$A^\alpha_l = A^\alpha \varphi_l \tag{3.7}$$

where φ_l is a covariant vector. Substituting in (3.3) and making use of (3.5), we find that φ_l is a gradient, and consequently (3.7) is of the form (3.2).

The case when equations (3.5) admit m ($<n$) sets of solutions, in terms of which any set of solutions is linearly expressible can be handled by a method similar to that used in §7 of the former paper. In this case any vector at a point P in the m-fold bundle of vectors determined by the m vectors at P is parallel to a vector in the corresponding bundle at any other point of the space.

4. In the preceding section we have given the conditions for one or more fields of vectors in invariantive form. Now we shall show how all such fields may be obtained by making a suitable choice of coördinates.

Suppose we have m ($<n$) fields of parallel vectors of components $A^i_{(p)}$, where $i = 1, \cdots n$; $p = 1, \cdots m$. If $B^i_{(p)}$ denote the components in another set of variables y^i, we have

$$B^i_{(p)} = A^\alpha_{(p)} \frac{\partial y^i}{\partial x^\alpha}. \tag{4.1}$$

If we show that there exist n independent functions y^i such that

$$X_p(y^i) \equiv A^\alpha_{(p)} \frac{\partial y^i}{\partial x^\alpha} = 0 \quad (i \neq p), \tag{4.2}$$

then in the new set of coördinates all the components of the vectors will be zero except those of the form $B^p_{(p)}$.

If we form the Poisson operator for (4.2), namely $(X_p X_q - X_q X_p)(y^i)$, we have in consequence of (3.1)

$$(X_p X_q - X_q X_p)(y^i) = A^\alpha_{(p)} \frac{\partial \log \varphi_q}{\partial x^\alpha} X_q(y^i) - A^\alpha_{(q)} \frac{\partial \log \varphi_p}{\partial x^\alpha} X_p(y^i),$$

where φ_p ($p = 1, \ldots m$) are the functions φ appearing in equations of the form (3.1). Hence (4.2) is a complete system.

If we let p take all the values from 1 to m, there are in accordance with the theory of complete systems $n - m$ independent solutions, which we take for $y^{m+1}, \ldots, y^n$. If we exclude from the system (4.2) the equation $X_r = 0$ where r has a value from 1 to m, we have a complete system of $m - 1$ equations, of which $n - m$ independent solutions are $y^{m+1}, \ldots, \ldots y^n$, and the other we take for y^r. Hence if there exist m fields of parallel vectors, the coördinates can be chosen so that all the components are zero except those of theform $A^p_{(p)}$ ($p = 1, \ldots, m$), and consequently our problem reduces to the determination of geometries for which equations (3.1) admit solutions of this kind.

5. In order that equations (3.1) admit solutions $A^1_{(1)} \neq 0$, $A^i_{(1)} = 0$ ($i \neq 1$), we must have

$$\Gamma^1_{1j} = 0, \quad \Gamma^i_{1j} = \frac{\partial}{\partial x^j} \log \psi \quad \begin{pmatrix} i = 2, \ldots n \\ j = 1, \ldots n \end{pmatrix}, \tag{5.1}$$

where ψ is an arbitrary function of the x's, and then $A^1_{(1)} = \varphi/\psi$. Consequently if we choose the Γ's with one or two subscripts 1 as given by (5.1), and take the others as arbitrary functions of the x's, we have the most general geometry with one field of parallel contravariant vectors.

In like manner any geometry with m fields of parallel contravariant vectors can be obtained by choosing

$$\Gamma^i_{pj} = 0 \quad \Gamma^p_{pj} = \frac{\partial}{\partial x^j} \log \psi_p \quad \begin{pmatrix} p = 1, \ldots m \\ i \neq p \\ j = 1, \ldots n \end{pmatrix}, \tag{5.2}$$

where ψ_p is an arbitrary function of $x^p, x^{m+1}, \ldots, x^n$.

6. Suppose now that the geometry is Riemannian, the fundamental form being

$$ds^2 = g_{ij}\,dx^i\,dx^j \quad (g_{ij} = g_{ji}) \tag{6.1}$$

Since

$$\Gamma^i_{ij} = \frac{1}{2}\,g^{i\alpha}\left(\frac{\partial g_{1\alpha}}{\partial x^j} + \frac{\partial g_{\alpha j}}{\partial x^1} - \frac{\partial g_{1j}}{\partial x^\alpha}\right),$$

equations (5.1) are in this case equivalent to

$$\frac{\partial g_{1\alpha}}{\partial x^j} + \frac{\partial g_{\alpha j}}{\partial x^1} - \frac{\partial g_{ij}}{\partial x^\alpha} = 2g_{1\alpha}\frac{\partial}{\partial x^j}\log\psi \tag{6.2}$$

for $\alpha, j = 1, \ldots n$. When we take $\alpha = j = 1$, we find $g_{11} = \psi^2\Phi(x^2, \ldots x^n)$. When we take $j = 1$, $\alpha \neq 1$, we find

$$\frac{\partial}{\partial x^1}\left(\frac{g_{1\alpha}}{\psi}\right) = \Phi\frac{\partial\psi}{\partial x^\alpha} + \frac{\psi}{2}\frac{\partial\Phi}{\partial x^\alpha}. \tag{6.3}$$

Consequently $g_{1\alpha}$ are given by quadratures, and likewise $g_{\alpha j}$ ($\alpha \neq 1$, $j \neq 1$)from (6.2).

In particular if ψ = const., by interchanging α and j in (6.2), we find that $\partial g_{\alpha j}/\partial x^1 = 0$ for all values of α and j. Moreover (6.2) becomes

$$\frac{\partial g_{1\alpha}}{\partial x^j} = \frac{\partial g_{1j}}{\partial x^\alpha}. \tag{6.4}$$

As a first consequence of this equation we have that g_{11} is a constant which may be taken equal to unity. Again (6.4) are the necessary and sufficient conditions that

$$g_{1\alpha}\,dx^\alpha = dx^1 + d\varphi\,(x^2, \ldots, x^n).$$

If then x^1 is replaced by $x^1 - \varphi\,(x^2, \ldots x^n)$, the form (6.1) becomes

$$ds^2 = dx_1^2 + g_{ij}\,dx^i\,dx^j \quad (i, j = 2, \ldots n), \tag{6.5}$$

where g_{ij} are independent of x^1. A space with linear element (6.5) is the most general which admits a translation into itself. (Bianchi, *Teoria dei gruppi continui*, Pisa, 1918, p. 500.) The space-time manifold of four dimensions used by Einstein in his cosmological considerations is of the type (6.5), x^1 being the coördinate of time.

In the case of $m(<n)$ fields of parallel vectors for which $\Gamma^i_{pj} = 0$, ($p = 1, \ldots m$; $i, j = 1, \ldots, n$), these equations are equivalent to

$$\frac{\partial g_{ji}}{\partial x^p} = 0, \quad \frac{\partial g_{pi}}{\partial x^j} = \frac{\partial g_{pj}}{\partial x^i}. \tag{6.6}$$

From the first of these it follows that all of the functions g_{ij} are independent of $x^1, \ldots x^m$. From the second of (6.6) we find that g_{pq} ($pq = 1,$

$..m$) are constants. Then as in the above case we show that the linear element can be put in the form

$$ds^2 = (dx^1)^2 + .. + (dx^m)^2 + g_{ij}\, dx^i\, dx^j \quad (i, j = m+1, .. n),$$

where g_{ij} are independent of $x^1, \ldots, x^m$. When $m = n - 1$, equation (6.7) is reducible to the euclidean form.

7. If A^i are the contravariant components of a vector in a Riemannian geometry, its covariant components A_i, are given by

$$A^i = g^{i\alpha}\, A_\alpha. \tag{7.1}$$

When this expression is substituted in (3.1), we obtain

$$\frac{\partial A_i}{\partial x^j} - \Gamma^\alpha_{ij}\, A_\alpha - A_i \frac{\partial \log \varphi}{\partial x^j} = 0. \tag{7.2}$$

When we are dealing with a non-Riemannian geometry we say that a field of parallel covariant vectors is one which satisfies (7.2).

The conditions of integrability of (7.2) are

$$A_\alpha\, B^\alpha_{ijk} = 0. \tag{7.3}$$

In order that the A's given by (7.3) shall satisfy (7.2) it is necessary that

$$A_\alpha\, B^\alpha_{ijkl} = 0. \tag{7.4}$$

As in the case of contravariant vectors, it can be shown that when there are m ($< n$) independent sets of solutions of (7.3) which satisfy (7.4) there exist m fields of parallel, covariant vectors.

The methods of §§ 4, 5 cannot be applied to the case of covariant fields.

ON THE INFLUENCE OF DENSITY OF POPULATION UPON THE RATE OF REPRODUCTION IN DROSOPHILA[1]

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It has long been known that degree of crowding of organisms in a given space, or the density of the population, has an influence upon various vital processes of the individuals composing the population. In the matter of growth Semper[2] and before him Jabez Hogg[3] showed that volume of water apart from food and other conditions has an influence upon the rate. This subject has again been studied recently by Bilski[4]. Farr[5] showed that there is in man a definite relation between density of population and the death rate. This old work of Farr's has recently been gone over carefully and confirmed by Brownlee.[6] Drzwina and Bohn[7] show that a particular concentration of a toxic substance, just lethal for a single

individual in a given volume of water (working with such organisms as infusoria, planarians, hydra, tadpoles, etc.), will be sub-lethal if several individuals are present in the same fixed volume of water.

Without undertaking at this time any systematic review of the literature on this subject it is our purpose in this paper to present some experimental data on the influence of density of population upon a different biological function than those mentioned above, namely specific reproductive rate. This variable we define as measured by the number of adults (imagoes) produced per mated female per day over a specified period. Our interest in this problem arose originally from purely practical considerations. Anyone doing experimental work with *Drosophila* has constantly arising this situation: he wishes to start an experiment with a large number of flies of homogeneous make-up as a group. What is the most expeditious way to get the desired numbers while still maintaining the requirement of homogeneity? Is it better to mate a considerable number of pairs in one bottle, to furnish the desired progeny, and if so, how many pairs? Or is it better to mate up one or two pairs only in each of several bottles? Which plan will give the largest progeny group? Not being able to answer this practically important question satisfactorily either by *a priori* reasoning, or by perusal of the *Drosophila* literature, it was decided to test it experimentally. The results proved to have a considerably wider interest than that implied in the purely practical question.

Plan of Experiment.—Mass matings were made on March 21, 1921 of flies from our line 107, the characteristics of which have been described by Pearl and Parker.[8] When the offspring from the matings emerged they were used to make up the matings of this experiment, according to the following plan. Half pint milk bottles were used, and the food arrangements, etc., were according to the standard *Drosophila* technique of this laboratory, fully described by Pearl and Parker.[9] The composition of the matings was as follows:

SERIES	NO. OF BOTTLES	MATED PAIRS IN EACH BOTTLE AT START	SERIES	NO. OF BOTTLES	MATED PAIRS IN EACH BOTTLE AT START
1	4	1	9	4	9
2	4	2	10	3	10
3	4	3	11	3	12
4	4	4	12	3	15
5	4	5	13	3	20
6	4	6	14	1	25
7	4	7	15	3	30
8	4	8	16	2	50

These bottles were put into a 25° incubator. At the end of 8 days the parent flies still surviving were transferred to fresh half pint bottles and allowed to continue breeding for a second period of 8 days.

All the progeny flies as they emerged as imagoes from the two series of bottles were counted and recorded as to sex. No account was kept of larvae or pupae. We are dealing here with net reproductive capacity, just as in certain of our poultry work (Pearl[10]). Food and temperature conditions were constant for all bottles throughout the experiment. The only significant variable between the several series was the density of population. There occurred various accidents to some of the bottles in the course of the experiment. Some of the mated flies died; food occasionally rose in the bottles and killed a few individuals, etc. In reducing the results to a per ♀ day basis allowance has been made for these accidents, and the final results set forth below represent the outcome for normal mated flies living under normal conditions.

Inasmuch as only 16 day's progeny was counted the results cannot be directly stated in terms of total reproductive capacity. Instead they have been tabulated as the rate of reproduction per ♀ day over a 16 day period after emergency. It is quite certain however that the absolute number of progeny per ♀ in the first 16 days of life is highly correlated with the unknown total reproductive capacity. This we feel sure of from many observations in *Drosophila* and from analogous observations on fecundity in the domestic fowl.

Results.—The results of the experiments are exhibited in Table 1, and graphically in figure 1. The table contains the following data: (*a*) the numbers of mated pairs per bottle at the start. The numbers of mated flies per bottle were therefore twice these figures at the start; (*b*) the mean population of mated flies over the whole period of 16 days. If no accidents had occurred these figures would have been 2, 4, 6, 8, 10, 12, 14, 16, etc. But accidents and deaths did occur. The figures set down represent the actual average number of flies present in each bottle over the period of the experiments, taken as a whole. (*c*) The mean number of flies per cubic centimeter of free air space in the bottle. Since all the bottles were by careful measurement of the food made to contain the same number of cc. (130) of free air space, it follows that the entries in this column are simply a constant ($^1/_{130}$) times the entries in the preceding column. They are inserted merely to show what the actual density in these experiments was. (*d*) The mean number of flies per square centimeter of agar, growing yeast on its surface, in each bottle. The same remarks apply to this as to the cubic density figure. It corresponds exactly to the density of population figure used in human vital statistics. The food area per bottle was 23.76 cm.^{2} (*e*) The total female days, meaning the number got by adding together for all females the number of days that each was in the experiment. Thus if a female lived throughout the experiment she contributed 16 ♀ days (1×16) to the total; if she lived 7 days she contributed 7 ♀ days (1×7) and so on. (*f*) Total absolute number of

imagoes produced in 16 days. These absolute figures vary of course with the accidental circumstances affecting the mated population. (*g*) The observed rate of reproduction, given as number of offspring imagoes per mated female per day. (*h*) The theoretical rate of reproduction (number of imagoes per mated female per day) as calculated from equation (*i*) below.

TABLE 1

EXPERIMENTAL DATA

PAIRS PER BOTTLE AT START	MEAN POP. AVERAGE 16 DAYS	MEAN FLIES PER CM² OF SPACE	MEAN FLIES PER CM² OF FOOD AREA	TOTAL ♀ DAYS	TOTAL IMAGOES IN 16 DAY	IMAGOES PER MATED♀ PER DAY	*y* FROM EQUATION (i)
1	2.00	.015	.084	63	1348	21.4	21.1
2	3.40	.026	.143	89	1124	12.6	14.5
3	5.50	.042	.231	173.5	1877	10.8	10.2
4	7.32	.056	.308	232	2100	9.1	8.1
5	9.29	.071	.391	267	1585	5.9	6.7
6	10.29	.079	.433	298.5	2034	6.8	6.2
7	11.51	.089	.484	377.5	1951	5.2	5.6
8	14.48	.111	.609	436.5	1996	4.6	4.6
9	16.62	.128	.700	519	2117	4.1	4.0
10	18.74	.144	.789	275	855	3.1	3.6
12	21.70	.167	.913	504	1746	3.5	3.1
15	26.02	.200	1.095	609.5	1389	2.3	2.5
20	34.02	.262	1.432	754	1666	2.2	1.8
25	47.75	.367	2.010	384	388	1.0	1.1
30	47.73	.367	2.009	1069.5	1272	1.2	1.1
50	90.66	.697	3.816	1446.5	474	.33	.34

It is at once apparent from this table that there is a profound and regular change in the rate of reproduction of *Drosophila*, under the conditions of these experiments, with increasing density of population. The rate of reproduction per mated female per day declines as density of population increases, at first extremely rapidly and then more and more slowly at higher densities. The total number of progeny flies (23,922) is great enough to give confidence in the results.

A better appreciation may be obtained of the nature and regularity of this change if we put the results in graphic form. This is done in Fig. 1.

The smooth curve passing through the circles is the graph of equation (i) fitted to the observations by the method of least squares. In this equation y denotes imagoes per mated female per day, and x denotes mean density of the mated population (measured as flies per bottle) over the whole 16 day period.

$$y = 34.53\, e^{-.018x}\, x^{-.658} \qquad \text{(i)}$$

Or, in logarithmic form,

$$\log y = 1.54 - .008\, x - .658 \log x \qquad \text{(ii)}$$

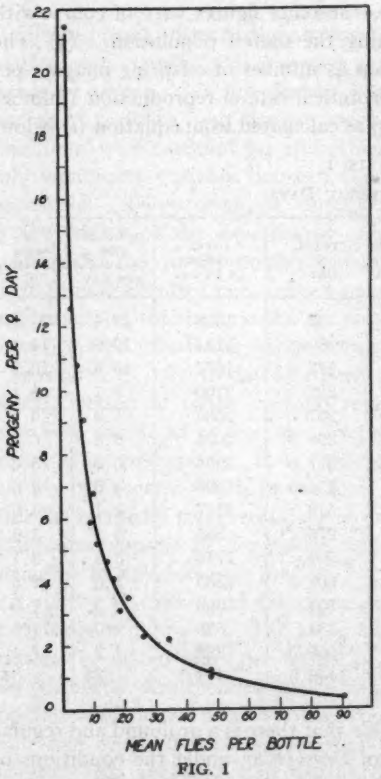

FIG. 1

Showing changes in rate of reproduction of *Drosophila* with increasing density of the mated population. The circles give the observations and the smooth curve is thegraph of the logarithmic equation discussed in the text.

It is at once apparent that this equation describes the observed facts with extraordinary precision. Rarely, even in a physical or chemical experiment, does one get such close agreement as is here shown between observation and theory. Plainly the curve is the expression of the law relating these two phenomena, rate of reproduction and density of population.

It is of interest to note that this equation is essentially identical in form with Farr's Law relating death-rate to population. Farr's original equation was the following, put in logarithmic form:

$$\log D = \log a + k \log d$$

where D denotes death-rate and d density of population and a and k are constants. Brownlee has slightly modified the constants but the essential form remains the same. Now in our equation (ii) the term in x is obviously of no practical importance, and could be left out entirely without visibly affecting the fit. It was only put in in a spirit of meticulous curve-fitting accuracy. With this term out our equation becomes in form identical with Farr's except that we have shifted the origin. The only essential difference is that in his case death-rate is shown to vary directly with density of population, while in the present case rate of reproduction varies inversely with density, but according to the same law. Death and reproduction are both fundamental and antithetical biological phenomena. What we know about the matter suggests that density of population has a profound effect upon many if not all general and fundamental biological processes. Brownlee points out that the same law holds in racing for the relationship between the record time for any definite length of course and the length itself, and for other biological and biochemical phenomena.

Bilski (*loc. cit.*) uses essentially the same equation to describe the influence of density of population and growth in tadpoles. Certainly the subject deserves further study. Possibly our present results offer a clue to a part at least of the solution of the problem of the practically world-wide decline in the birth-rate which had been going on for more than a quarter of a century before the outbreak of the war.

Discussion.—How can the result be explained? The extreme closeness of the observations to the theoretical curve indicates clearly that we are dealing here with no haphazard performance, but with a highly lawful phenomenon. The first point that occurs to one is that since we are dealing with imagoes only, the result may perhaps be explained by larval and pupal elimination due to overcrowding in these stages. In the bottle say with 20 females there were laid, on this view, something like 20 times as many eggs as in the bottle containing one female, these eggs developed into 20 times as many larvae, there was not room or food for so many larvae, therefore many of them were selectively eliminated, and the survivors which got through to the imago stage indicated only a low reproduction rate, which represented however only the capacity of 80 cc. of banana-agar or 23.76 cm.^{2} of food surface to nourish larvae, and nothing else.

Plausible as this hypothesis is—the more so because it would at once indicate why our rate of reproduction variable follows Farr's Law, since it would mean Farr's Law applying to the larvae—it cannot possibly account for the facts. To demonstrate this we have only to examine with some care the absolute figures given in the fifth column of Table 1. In the first place let it be recalled that the greatest drop in rate of reproduction (imagoes per ♀ per day) occurs as we pass from a mean density of 2 flies per bottle to one of 3.4 flies per bottle (1 and 2 pairs originally). The next greatest drop in the rate is from the bottles of density 3.40 to those of density 5.50. Now at all these low densities one in fact cannot possibly speak of larval crowding at all. This anyone knows who has experimentally bred *Drosophila*. A half-pint bottle with 80 cc. of banana-agar will support many more than all the larvae one or two or three females will produce in 8 days. This is indicated by the absolute figures in the fifth column of Table 1. The bottles containing 9 mated pairs at the start produced 2117 offspring imagoes. Therefore the food was sufficient to allow at least that many larvae to come through successfully. But in the bottles with 1, 2, or 3 mated females the absolute number of larvae was much less than this. Now the drop from the one pair to the two pair bottles in rate of reproduction was 8.8. To prove that this cannot be explained by larval crowding and elimination we have only to multiply 21.4 the rate per ♀ day in the 1 pair bottles, by 89 the number of ♀ days in the 2-pair bottles. This will give the number of imagoes which would have been produced in the 2-pair bottles had the reproduction rate of the

1-pair bottles held there. The result is 1904.6. *But this is 200 or more less than the absolute number produced in various other bottles.* Therefore the drop in the rate from the 1-pair to the 2-pair bottles cannot be explained on the supposition of larval crowding and elimination. Possibly this factor comes into play in the higher densities, though there is no evidence as yet that such is the fact.

That density of population influences fecundity in the same sense that it is here shown to affect rate of reproduction was demonstrated with fowls more than a decade ago.[11] It was found that there was a steady decline in mean annual fecundity in Barred Rock pullets as one increased the number of birds per year from 50 to 100 and to 150. The experiments were so arranged that the square feet of floor space per bird (that is to say the density of population in Farr's sense) was substantially identical in all three sets—exactly so in the case of the 50 and 100 bird pens, and only slightly less so in the 150 bird pens. We cannot go into the details of these careful and critical experiments with poultry here, but would point out that they indicate that there is apparently another element involved in the case besides mere physical density, which element in our ignorance we may perhaps tentatively call psychological. Apparently there is a physiological effect of keeping large numbers of individuals together in a confined area or space, even though the amount of space or area *per individual* is identically the same in the large crowds of individuals as in the small crowds.

In general there can be no question that this whole matter of influence of density of population, in all senses, upon biological phenomena, deserves a great deal more investigation than it has had. The indications all are that it is the most important and significant element in the biological, as distinguished from the physical, environment of organisms.

Summary.—In this paper it is shown that the rate of reproduction of *Drosophila* during the first 16 days of its imaginal life varies inversely with density of population, in an extremely exact manner, in accordance with the following equation which is essentially an inverse form of Farr's Law,

$$\log y = a - bx - c \log x.$$

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University. No. 57.

[2] Semper, K. *The Natural Conditions of Existence as they Affect Animal Life.* Fourth Edit. London 1890. (Other references to detailed papers are given in this book.)

[3] Cited from Semper, *loc. cit.*

[4] Bilski, F. *Pflüger's Arch.* **188,** pp. 254–272, 1921.

[5] Farr, Wm. *Decennial Suppl. Reg. Gen.* 1861–70.

[6] Brownlee, J. *J. Roy. Stat. Soc.* **82,** pp. 34–77, 1919, and **83,** pp. 280–283, 1920.

[7] Drzwina, A., and Bohn, G., *C. R. Soc. Biol. Paris,* **84,** pp. 917–919, 1921.

[8] Pearl, R. and Parker, S. L. *Amer. Nat.* **56,** pp. 174–187, 1922.

[9] Pearl, R. and Parker, S. L. *Amer. Nat.* **55**, pp. 481–509, 1921.
[10] Pearl, R. *Genetics*. **2**, pp. 417–432, 1917.
[11] Pearl, R. and Surface, F. M. *U. S. Dept. Agr. Bur. Amer. Ind. Bulletin* No. 110, Part 1, pp. 1–80, 1909.

ANIMAL EVOLUTION

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There are few lines of scientific speculation of such general interest as the problem of the evolution of the varied assemblage of types included in the animal kingom, while at the same time a logical and detached consideration of this problem is rendered exceedingly difficult both because of the isolated position occupied by many of these types and by the conviction we all have that man must represent the highest of them.

But while undoubtedly man is the most efficient and the dominant member of the animal kingdom it by no means necessarily follows that he and his fellow vertebrates are from the strictly biological standpoint the most perfect.

In the following pages I shall indicate, as briefly as possible, a line of reasoning whereby all the various animal types are brought into correlation with each other, and their evolution is shown to be not evolution in the sense of a progressive development from a lower type to a higher, but instead the gradual acquisition of increasing economic efficiency through the progressively greater and greater departure from biological perfection, correlated with the gradual loosening of the bonds by which the most perfect type is economically handicapped.

In a very large and important group of animals, the Protozoa, the body is composed of a single cell; but in all animals except the Protozoa the body is composed of a very large number of cells which are differentiated to serve definite purposes.

All multicellular animals begin life as a single cell. As at this stage they are in this respect comparable in structure to a protozoan, in later life they may be assumed to represent an advance over the protozoan type.

The original cell giving rise to a multicellular animal typically divides into two, four, and eight similar cells, the cleavage taking place in three planes each at right angles to the other two; each of these eight similar cells continues to divide until a hollow ball is formed of numerous cells all of which are alike. Such a structure is represented by a blastula, but no adult animal (? except *Salinella*) which can be regarded as the equivalent of such a stage is known.

The hollow sphere with its increasingly thin walls due to repeated cell divisions becomes mechanically unstable and collapses, forming a cup with a two layered wall, the inner lining being composed of the inturned cells of the collapsed portion.

There is now a differentiation of the cells into an external and an internal layer, but each of the cells in both layers is the equivalent of all the others in the same layer. A definite axis has appeared, passing through the center of the mouth of the cup (the gastrula opening) and through the opposite pole.

Since in every plane passing through the central axis the body wall is just the same as it is in every other plane the continued development of such a typical gastrula would result in the formation of a radially symmetrical animal developed about the axis of the gastrula as a center.

The logical conclusion is that those animals which develop radially about the original axis are biologically the most perfect of all animals since they continue without interruption into the adult stage the original developmental plan.

There are two groups of animals which develop in this way. In one of these, the sponges, the larva grows into a sort of radial mass colony or imperfectly integrated community of cells with little division of labor or unified life and no definite organs. In the other, the cœlenterates, the gastrula grows into an adult which resembles it in all its essential features, though innumerable minor refinements are added.

The sponges, being without any definite structures, indeed little more than proliferated masses of slightly differentiated cells, increase in size by simply growing out in all directions from the main axis and usually also upward.

The cœlenterates retain the definite body form of the original gastrula which with increase in size rapidly weakens mechanically in spite of the development of a series of internal buttresses; furthermore, they have definite organs, such as tentacles, which must be supported by the body wall, and they must be strong enough to capture and to hold the organisms which serve them as food. They are therefore limited to a definite and relatively small body size, excepting in a few cases of extreme specialization. But just as in the sponges there is no definite limit to the original growth impetus. In response to this the original polyp, having reached its full size, gives off a bud which grows into another polyp and this process may be continued indefinitely so that enormous arborescent or other colonies are formed.

Both the sponges and the cœlenterates continue in the same way to increase in bulk throughout life, the former by simple augmentation of the mass as a whole, the latter by the indefinite reduplication of the original unit. The colonial habit is a fundamental attribute of all cœlenterates,

though many of them have secondarily become solitary through the disintegration of the colony coupled with an increase in size of the individuals.

Among the cœlenterates the solitary habit is commonly derived from the colonial through the process of strobilization. The original polyp elongates and from it plate-like units (ephyræ) one by one separate off by transverse fission distally which develop into large free living adults.

The original polyp which gives rise to the strobila is always at first bilaterally symmetrical, with two tentacles opposite each other. In the simpler cœlenterates the gastrula becomes the mouth of the adult, but in the more specialized types a considerable part of the external surface about this opening is drawn into it during growth forming a gullet tube.

Bearing these two facts in mind it is easy to reconstruct the path by which the so-called higher animals were evolved from the cœlenterate type.

Strobilization in the form characteristic of the cœlenterates became modified by the retention of the original bilateral symmetry and its accentuation, coupled with a more or less vague differentiation into a dorsal and a ventral surface, and the retardation of the liberation of the units formed, which are not detached until after sexual maturity has been reached, and furthermore have to a considerable degree lost their individuality, sharing a common nervous and excretory system. This condition is represented by the tape-worms, at the present time occurring only as parasites.

The next step was the further reduction in the individuality of the segments and their retention throughout life, resulting in the formation of a definite elongate jointed body composed of a series of similar segments in which a dorsal and a ventral surface are clearly distinguishable. Such a condition is represented by the annelids.

The unified body thus formed by the consolidation of what were originally separate and distinct entities in an animal colony which have now quite lost their individuality and become mere segments was still further perfected by the grouping of these segments into three units, the head, most unified, controlling and directing, the thorax, less completely unified, locomotor, and the abdomen, largest and least unified, containing the vital organs, as seen in the insects, or by a marked tendency to compress the body within the compass of a single group of firmly consolidated units, as seen in the crustaceans. In both cases the consolidation of the body was accompanied by a marked increase in the differentiation of the dorsal and ventral surfaces, and an extension of the former at the expense of the latter.

The body of the insects and crustaceans was thus derived from the colonial cœlenterates (as represented by the strobila) through the cestode and annelid types by the reduction of the colonial units to segments with progressively decreasing individuality accompanied by the modification of the

radial symmetry first to a symmetry in which the body is the same on either side of two planes at right angles to each other (as in the polyp giving rise to the strobila), then to a similar but much more pronounced symmetry of the same kind with a faintly indicated dorsal and ventral surface (as in the cestodes), later to a bilateral symmetry with well differentiated dorsal and ventral surfaces, the former usually somewhat more extensive than the latter (as in the annelids), and finally to a very pronounced bilateral symmetry with the dorsal surface always greatly in excess of the ventral (as in the crustaceans and insects). The two tentacles of the polyp from which the strobila develops persisted as the lateral extensions of the proglottides in the tape-worms (which sometimes reach a considerable length), and these reappeared as the parapodia of the annelids, and in their most perfected form as the legs and other segmental appendages of the arthropods.

Strobilization comparable to the type characteristic of the cœlenterates also occurs in another form. The units, instead of budding off in a linear series, are formed on the walls of the interior of a closed polyp which finally disintegrates, liberating all the enclosed units as free living individuals. These have a symmetry which is apparently bilateral, but always shows marked radial features. Such a development is found in the liver fluke and its allies, though today in no non-parasitic forms.

This method of growth by which young are actually seen to be budded off from the interior of a closed sack developed from the ciliated larva, combined with the more or less extensive infolding within the gastrula mouth of part of the outer body wall as seen in the more specialized cœlenterates, furnishes a clue to the origin of the unsegmented animals.

In those colonial cœlenterates with division of labor the polyps are divided primarily into nutritive, reproductive, and "defensive," but the last are characterized by an abundance of cells containing an excretion and a coiled tubule, and some or other of them are constantly discharging the included liquid. They are probably fundamentally excretory and only secondarily adapted for defensive purposes.

It is easy to imagine that in the early stage of some cœlenterate type the infolding of the outer cell wall at the gastrula mouth was sufficient to bring within the inner cavity the region from which the new polyps are budded off, and to assume that there then were given off internally three buds, one of the nutritive (sack-like), one of the reproductive, and one of the excretory type, which developed into the perivisceral, the gonadial, and the nephridial cœlome.

The so-called cœlomate animals with unsegmented bodies, none of which, in sharp contrast to the segmented animals, have any true appendages, thus represent a colonial cœlenterate in which the colonial development has been transferred within the original unit; and the development of

this internal colony has reached its maximum in the cephalochordates and the vertebrates.

The correctness of the interpretation of the segmented animals as primarily consolidated colonies is indicated by the retention in all the included groups—the cestodes, annelids, crustaceans, echinoderms and insects—of some form of asexual reproduction, budding, budding and fission, fragmentation, polyembryony or parthenogenesis which, though greatly reduced in the more specialized types, especially in the insects, may be traced back to the budding and budding and fission characteristic of the cœlenterate colonies.

The correctness of the interpretation of the unsegmented animals (except those with gill apertures) as ingrown colonies is indicated in all the included groups—the priapulids, sipunculids, molluscs, nemerteans, phoronids, brachiopods, and chætognaths—by the entire absence of asexual reproduction of any kind, this having been rendered impossible through the transference of the colonial development to the interior.

For the sake of clearness no mention has been made of the fact that the annelids, crustaceans, echinoderms and insects, belonging to the segmented series, also possess cœlomic structures and are to a certain extent therefore doubly colonial. In the annelids the two structural remnants of the primitive colony are about equally balanced. In the crustaceans and insects the cœlomic structures have almost disappeared in favor of the development of the segmentation. In the echinoderms the body is reduced to five half segments while the cœlome is highly developed.

With the segmentation and the development of the cœlome considered as derived from a fundamentally colonial habit there is no fundamental body form nor structure in the bilateral animals which cannot be traced back to an origin in the colonial cœlenterates. The cestodes and the trematodes represent the transition forms in which we see the colonial habit and the radial symmetry of the cœlenterate type breaking down and passing into the solitary habit and bilateral symmetry of the "higher" animals in which little or no traces either of the primitive radial symmetry or of asexual reproduction remain.

These numerous forms, the annelids, crustaceans, echinoderms and insects on the one hand and the priapulids, sipunculids, molluscs, nemerteans, phoronids, brachiopods, chætognaths, enteropneusts and vertebrates on the other are so inextricably entangled in a network of lines connecting each more or less definitely or vaguely with all the others and even crossing the gap between the two groups that it is quite beside the mark to speak of evolution in connection with them. Each phylum represents a sort of crystallization center about which a greater or lesser number of forms all showing but little deviation from a fixed type are closely grouped within a definite but broad and general structural complex.

Some of these crystallization centers or foci lie close together, as the crustaceans and the insects, while others are quite isolated, like the nemerteans, phoronids, brachiopods, chætognaths and echinoderms. They are not now connected by any intermediate forms, and there is no evidence that they ever were. Each represents a stable and economically efficient, circumscribed and distinctive, combination of structural characters crystallized out of a wide range of possibilities resulting from the disintegration of the cœlenterate type which has, so to speak, gone into solution.

Evolution, in the sense of progressive specialization, has been conclusively demonstrated within many groups, but always within restricted sections of these groups. There is no evidence that the larger groups, or the broader divisions within these groups, succeeded each other in any such way. The appearance of all the major multicellular animal types was probably simultaneous or nearly so, the original progenitor giving rise continually to innumerable variations in form, symmetry and organization most of which proved economically impossible, but many of which, proving suitable to meet a certain range of conditions, formed centers about which the various phyla were developed.

Whereas the animals between the cœlenterates and the enteropneusts are commonly considered as representing progressive evolution, this is not so in fact. It is true that they represent increasing bodily efficiency, but this bodily efficiency results from progressive structural deficiencies which throw into relief the features upon which this bodily efficiency is based. All the essential features of the bilateral animals preëxist in some form or other in the cœlenterates, but they can only become effective economically through the partial or more or less complete inhibition of others which here hold them in check.

Unimportant as they seem to most morphologists the possession of gill apertures characterizes a most interesting group of diverse animal types, the pterobranchiates, balanoglossids, tunicates, cephalochordates and vertebrates. In all of these except the balanoglossids there is a sudden return to the fundamental colonial habit in a modified form, as if the appearance of gill apertures marked the beginning of the senescence of the animal line as a whole which from that point on recapitulated, or rather paralleled, its earlier stages. In the cephalochordates and vertebrates the internal colony or cœlome becomes enormously subdivided, recalling more or less closely the conditions in the annelids; on the other hand in the tunicates there is practically no cœlome, but the formation of colonies by budding, which are often highly complicated and with division of labor, reappears. In the pterobranchiates (*Rhabdopleura* and *Cephalodiscus*) the formation of colonies by asexual budding is developed to a high degree.

From the preceding discussion it would seem that the only way by

which the numerous and varied animal types can be made to fall into any evolutionary plan is to consider the radially symmetrical colonial cœlenterates as representing the highest degree of purely biological perfection, and all other animals (except the sponges which are their equivalent) as derived from them through the appearance of various defects which had the anomalous result of leading to increased bodily efficiency.

THE CORRELATION OF TIME UNITS AND CLIMATIC CHANGES IN PEAT DEPOSITS OF THE UNITED STATES AND EUROPE

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This paper is a continuation of a previous publication[4] which contained a discussion of the evidence in peat deposits of the United States regarding climatic changes since the last Ice age.

For purposes of comparison a brief account in tabular form is presented to show certain correlations between peat deposits in this country with those of the continent of Europe.

1. A correlation of American with European *glacial drift sheets* has been made by Leverett.[8] By considering with Leverett the early and late Wisconsin Glaciation contemporaneous with the Würm glaciation it is possible to correlate the corresponding minor glacial stages.

The three readvances of the ice sheet that produced the morainic systems of the late Wisconsin: the Valparaiso-Kalamazoo, Lake Border, and Port Huron may be regarded as corresponding to Penck and Brückner's[10] cycle of mountain glaciations of the Alps: the Bühl, Gschnitz, and Daunstadia—and to those of the Scandinavian continental ice-fields: DeGeer's[7] Dani-, Goti-, and Fini-glacial stadia.

2. *The Stratigraphic Study of Peat Deposits* in the United States has furnished valuable data as to the botanical composition of types of peat and their economic value. It has been shown also that the material formed under wet conditions differs in botanical composition and texture from that formed during dry periods, and that the study of separate and distinct layers in deposits formed within the main morainic systems during the glacial retreat serves to throw much light on the character of the past climatic changes.

Evidence is now presented by means of the position of conspicuous layers of forest peat, fibrous sedge and reed peat, and colloidal peat, to show the equivalence of peat deposits which were formed in the respective morainal belts of the United States and Europe. The greater structural complexity, it will be noted, is confined to areas within the oldest of the morainic

systems, while within the later and more recent positions of the ice border the deposits are younger in age and consist of only a few layers of peat. The lack of structural diversity in the uppermost layers of the older deposits is related probably to the distance of those deposits from the direct climatic influence of the later glacial sub-stages.

TABLE I

CORRELATION OF GLACIAL, CLIMATIC, AND LIFE STAGES SINCE THE LAST ICE-AGE IN THE UNITED STATES

GLACIAL SUB-STAGES	GREAT LAKES STAGES	CLIMATIC CHANGES	ESTIMATED TIME	ANIMAL REMAINS	HUMAN REMAIN	VEGETATION SUCCESSION
8. Present	Lake Michigan 580 ft.		A.D.			Forest Spagnum bogs and marsh
7. Recent (post-glacial)	Post-Nippissing	Rising dry	—o— B.C.			
						Forest
	Nippissing 530–590 Champlain Sea, Niagara	Temperate, humid			Mound builders?	Marsh
6. Port Huron	Algogum 608–609		7000–5000 B.C.			
morainic system	Lundy-Duluth 615 ft.	Temperate		Mastodon		
		Dry				Forest
5 Lake Border	Whittlesey-Chicago-Saginaw	Cool, humid			Flint implements	Marsh
morainic system	Maumee 805 ft.	Warm, dry	11000–8000 B.C.			
						Forest
4. Valparaiso	Late Wisconsin drift	Warm, humid	18000–16000 B.C.	Mastodon		Marsh
Kalamazoo morainic system	Loess formed	Cool, windy, dry				Steppe?
		Temporary amelioration				Low shrubs?
3. Bloomington		Cold, humid	25000 B.C.			
morainic system 2. Shelbyville moraines		Cold	28000 B.C.			Tundra?
1. Wisconsin driftborder		Arctic	35000–30000 B.C.			

To the practical worker it is obvious that a better understanding of the structural differences in peat deposits implies a more correct estimate of their economic value.

The profile sections visualize the stratigraphic relationship of American and European peatland and they suggest the value of a more detailed

TABLE II

STRATIGRAPHIC FEATURES OF PEAT DEPOSITS IN THE U. S.

	I	II	III	IV	V	VI	VII	VIII	IX	X
	CANTON, OHIO	KENT, OHIO	KANKAKEE, IND.	DISMAL SWAMP, VA.	EVERGLADES, FLA.	MANTUA, OHIO	FREDMONT, IND.	SANILAC, MICH.	ROME, N.Y.	CHASSEL, MICH.
8.	Sedge	Forest	Forest	Forest	Sedge	Forest	Sedge	Shrub	Forest	Forest
7.	Sedge	+ Sphagnum	+ Macerated	+ Macerated	+ Macerated		+ Macerated	Sedge	+ Sedge	+ Sphagnum Sedge
—	— — —	— — —	— — —	— — — —	— — —	— — —	— — —	— — —	— — —	— — —
—	?	Sedge	Sedge-Reed	Forest	Sedge	Forest	Colloidal	Forest	Forest	Forest
—	— — —	— — —	— — —	— — — —	— — —	— — —	— — —	— — —	— — —	— — —
6.	Sedge + Hypnum	Macerated	Sedge Macerated	+ Macerated	+ Macerated		+ Macerated	Sedge	Hypnum	
—	— — —	— — —	— — —	— — — —	— — —	— — —	— — —	Macerated	— — —	— — —
	Sedge	Forest	Shrub + Sedge	Forest	Sedge	Forest	Colloidal			
—	— — —	— — —	— — —	— — — —	— — —	— — —	— — —	— — —	— — —	— — —
5.	+ Hypnum	Sedge	Sedge	+ Macerated	+ Macerated	Sedge	+ Macerated Sand			
—	— — —	— — —	— — —	— — — —	— — —	Macerated	Macerated			
—	Reed	Forest	Forest	Forest	Sedge	— — —	— — —	— — —	— — —	— — —
4.	Macerated Clay	Sedge Macerated				— — —	— — —	— — —	— — —	— — —
—	Macerated	— — —	— — —	— — — —	— — —	— — —	— — —	— — —	— — —	— — —
—	— — —	— — —	— — —	— — — —	— — —	— — —	— — —	— — —	— — —	— — —
3.										
2.										
1.										

study of peat utilization in Europe in order to determine the best practices necessary to a successful solution of the peatland problem in this country on related peat deposits.

3. The common *time* relationship is confirmed by a series of correlations. It is based on the similarity between the groups of morainal systems as the great, basic divisions of time, and on the loess deposits resting upon the

TABLE III

CORRELATION OF GLACIAL, CLIMATIC, AND LIFE STAGES SINCE THE LAST ICE-AGE IN EUROPE

GLACIAL SUB-STAGES	BALTIC LAKE STAGES	CLIMATIC CHANGES	ESTIMATED TIME	ANIMAL REMAINS	HUMAN* REMAINS	VEGETAION SUCCESSION
8. Present	Baltic o.m.	Rising, dry, cool, moist	A.D. —o— B.C.			Forest
7. Recent (Post-glacial)			1800 B.C.–300 A.D.		Iron age	Younger sphagnum bogs
		Warm, dry	3000–1800 B.C.			Forest Heath (beech, fir)
	Litorina sea	Warm, moist	4000–3000 B.C.		Shell heaps	Older
6. Daun	0–22 m.		7000–5000 B.C.		Neolithic	sphagnum
Fini-glacial		Continental			culture Lake dwellers	bogs
					Azilian	Forest (oak)
5. Gschnitz			10000–8000 B.C.		Paleolithic	
Goti-glacial	Ancylus Lake 66–175 m.				culture	Marsh
						Forest (birch, pine)
4. Bühl		Severe Continental	18000–16000 B.C.	Steppe horse	Magdalen	Marsh Steppe
Dani-glacial	Yoldia sea 300 m.	Loess formed		Reindeer	culture	Dryas Steppe
		Temp. amelioration, Warm, arid				Low shrubs (Aspen willow)
3.				Mammoth Reindeer	Solutrean	Sub-arctic
2.					culture Aurignacean	Tundra
1. Wurm drift border		Arctic	30000 B.C.		culture Mousterian culture	

*Data taken in large part from Osborn (9)

moraines of the Würm-Bühl interstadium and the early Wisconsin drift, on the glacial lake stages of the Great Lakes and the Baltic, on the series of climatic changes, and on the remarkable association of the layers of peat material in the deposits which are referable to the same glacial substage; they mark the lesser divisions of time in the different geographic regions. These events were not local phenomena but well-marked stages

which occurred nearly simultaneously over a large part of northern America and northern Europe. The combined evidence suggests that it is possible to date with fair exactness the general course of events since the last Ice-age. This chronology may furnish important data bearing on different branches of science.

(*a*) The estimate on the duration of Post-Wisconsin glacial time in the United States is based chiefly on the recession of the waterfalls of the Niagara. The figures for the larger subdivisions fall within the estimates made by Reeds,[12] Coleman,[3] Taylor,[8] Chamberlain and Salisbury[2].

TABLE IV

PROFILE SECTIONS FROM PEAT DEPOSIT OF EUROPE

	XI	XII	XIII	XIV	XV	XVI	XVII
	GERMANY	ENGLAND	AUSTRIA ALPS	SWEDEN	SWEDEN	RUSSIA	RUSSIA
8. 7.	Sphagnum	Top removed Sphagnum	Forest + Sphagnum	Sphagnum	(Forest) + Sphagnum	(Forest) + Sphagnum	(Forest) + Sphagnum
	Heath	?	Shrub forest	Forest shrub	?	Forest	Forest
6.	Sphagnum	Sphagnum	Sphagnum	Sedge-Reed	Sphagnum	Sedge	Sphagnum
	Sedge or Forest shrub	Forest	Forest	Reed	Forest shrub	Forest	Hypmum Reed
5.							Sedge
	Sedge + Reed	Sedge + Reed	Reed	Macerated	Sedge Macerated	Macerated	
	Reed or forest	Reed and drifted timber	Reed	Macerated			
4.	Macerated	Hypnum	Hypnum				
	Macerated	Macerated					
3.							
2.							
1.							

(*b*) The time units of measurement of Penck,[10] DeGeer,[7] Wahnschaffe[15] and Brooks[1] are more or less similar with the dates delimiting the American stages.

4. The correlation of past *climates* in the United States with those of the European last glacial period has not been worked out on a meteorological basis but there is sufficient information at hand of considerable interest for comparison with the conclusions based on European evidence.

The results of the correlation agree in showing that, although the evolution of climate has been continuous, there have been several climatic waves since the last glacial period, each of relatively long duration. The climatic pulsation, resembling a climatic belt, consisted of a cold and humid period bordering the glacial readvance, followed by a warmer, drier period

in the accompanying interstadial time, with a tendency toward optimum conditions which favored the growth of forests. It is quite probable that the last climatic wave or belt, possibly in diminishing intensity, continues to shift northward at the present time.

5. In the past sixty centuries the power of nations has been moving rather steadily northward, and the higher production of agricultural crops has occurred in general near the northern limits of distribution. It is significant that this northward tendency in agricultural and social development seems to be continuing and that it appears to be correlated with practically the same course of events which is shown in Tables I to IV.

PROFILE SECTIONS OF PEAT DEPOSITS

I, II, VI and VII from *Botanical Gazette*,[5] **72**, 57–89, 1921. (4)

III. Profile of Kankakee Marsh near Crumstown, Indiana.

IV. Cross Section of Dismal Swamp, Virginia, generalized from soundings and exposures along the feeder ditch and the Dismal Swamp canal.

V. Cross section of Torry Island in Lake Okeechobee, Florida, also near Okeelanta, about five miles south from the lake shore.

VIII. Sounding on peat experiment sub-station near Sanilac, Michigan.

IX. Profile along barge canal near James street bridge at Rome, N. Y.

X. Sounding in central location of peat deposits at Chassel-Klingville, Michigan.

XI. After Weber. (16) XII. After Rankin. (11)

XIII. After Shreiber. (14) XIV. After Sernander. (13)

XV. After Erdtman. (6) XVI, XVII. After Docturowski. (5)

[1] *Brooks, C. E. P.* Evolution of climate in North-west Europe. *Quar. J. Roy. Meteorol. Soc.*, **47**, 1921 (173–194).

[2] *Chamberlin, T. C. & Salisbury, R. D.* The pleistocene or glacial period. *Geology*, Vol. 3, 1906.

[3] *Coleman, A. P.* Length and character of the earliest interglacial period. *Bull. Geol. Soc. America*, **26**, 1915 (243–245).

[4] *Dachnowski, A. P.* Peat deposits and their evidence of climatic changes. *Chicago, Bot. Gaz.*, **72**, 1921 (57–89).

[5] *Docturowski, V. S.* Vidy torfa. (Les espèces de tourbe) *Viestnik torfianovo diela*, **2**, 1915 (273–304).

[6] *Erdtman, O. G. E.* Pollenanalytische Untersuchungen von Torfmooren und Marinen Sedimenten in Südwest-Schweden. *Arkiv. Bot.*, **17**, 1921, No. 10, (73–79).

[7] *Geer, G. de* A geochronology of the last 12000 years. *11 Internat. Geol. Kong. Stockholm*, 1910, 1912 (241–253).

[8] *Leverett, F. and Taylor, F. B.* The pleistocene of Indiana and Michigan and the history of the Great Lakes. *U. S. Geol. Survey, Monographs*, **53**, 1915 (529).

[9] *Osborn, H. F. Men of the Old Stone Age*, 1916.

[10] *Penck A. and Brückner, E.* Die Alpen im Eiszeitalter, Leipzig. 3 vols. 1901–1909.

[11] *Rankin, W. M.* The lowland moors (mosses) of Lansdale (North Lancashire) and their development from Fens. *Types of British Vegetation*, 1911 (256–259).

[12] *Reeds, C. A.* Graphic projection of Pleistocene climatic oscillations. *Bull. Geol. Soc. of Amer.*, **26**, 1915 (106–109).

[13] *Sernander, R.* On the evidences of Postglacial changes of climate furnished by the peat-mosses of Northern Europe. *Geolog. Föreningens förhandl., Stockholm*, **30**,

1909 (465–473), also *Congrèss Géologigues International, 11 Sèssion, Stockholm*, 1910, **1**, 1912 (203–211).

[14] *Schreiber, H.* Nacheiszeitliche Klimaänderung und Moorbildung in Skandinavien. *Oesterreicshische Moorzeitschrift*, **15**, 1914 (104).

[15] *Wahnschaffe, F.* Über die Gliederung der Glazialbidungen Norddeutschlands und die Stellung des norddeutschen Randlosses. *Zs. Gletscherk*, **5**, 1911 (321–338).

[16] *Weber, C. Z.* Was lehrt der Aufbau der Moore Norddeutschlands über den Wechsel des Klimas in post glacialer Zeit? *Zs. Deut. Geol. Gesell.*, **62**, 1910 (143–162).

THE SPECTRAL ENERGY DISTRIBUTION AND OPACITY OF WIRE EXPLOSION VAPORS

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It has been shown[1] that a fine wire, through which is passed the discharge of a large high voltage condenser, vaporizes so rapidly that the phenomenon is best described as an explosion. The actual time required for the wire to pass from the solid to the vapor state is probably less than one millionth of a second. If the explosion is made to take place in the space between two parallel planes from 2 to 3 millimeters apart, such as a slot in a block of wood, its spectrum is continuous, the lines of the metal exploded appearing as absorption lines. The maximum intensity of the light is very great, being of the order of that to be expected from a black body at a temperature of about 20,000° K.

A preliminary study of the spectral energy distribution given by the explosion of iron wires has been made, using a quartz spectrograph with a vacuum thermo-couple[2] and galvanometer. The value of $E\lambda d\lambda$ increases very rapidly from the infra red to about λ4300; from this point to the ultra violet the great groups of iron absorption lines cause the values of $E\lambda d\lambda$ to oscillate considerably but its general tendency is upwards and the highest values are reached between λ2600 and λ2150. From λ2150 to λ1990 the values fall off rapidly, in part, perhaps, due to the absorption of the quartz lenses and prisms, and to that of the air.

Spectrograms of the explosions of the following metals have been made using a one meter focus grating spectrograph: copper, silver, gold, magnesium, zinc, cadmium, aluminum, tin, lead, tungsten, iron and nickel. All of these give continuous spectra of the same general intensity most of them appearing rather more intense than iron on account of the numerous absorption lines of the latter. Copper, silver and gold are anomalous in that the pure wires of these metals will not explode properly when placed in a slot in a block of wood, the main discharge always seeking a path around the

outside of the wooden block. By very slightly amalgamating the surface of the wires they explode normally, giving fine continuous spectra crossed by their own absorption lines, the lines due to Hg not having been observed.

That a layer of these metallic vapors only a few centimeters thick should give a continuous spectrum is somewhat surprising, especially so, since the average pressure is certainly much lower than it was earlier[3] thought to be. Professor Henry Norris Russell suggested that it would be of some interest to find out whether the vapors are transparent to radiation or not. The following experiment was accordingly performed: A short spark gap connected in series with the wire to be exploded was (*a*) placed immediately in front of the wire, so that the light from the explosion had to pass through the spark on its way to the spectrograph; (*b*) placed immediately back of the wire, so the light from the spark had to pass through the explosion vapors before reaching the spectrograph. The spark and explosion would be exactly simultaneous, since they were connected in series. Iron wires were used, and the spark terminals were made of brass in order to make use of the bright emission lines of zinc throughout the spectrum and those of copper in the extreme ultra violet.

Using the arrangement (*a*) the spectrogram showed the bright zinc and copper lines very distinctly superposed on the regular iron absorption spectrum. It was also clear that the iron lines were less dark than usual, that is they were partially filled up by the continuous background in the light from the spark. With the arrangement (*b*) no trace of the zinc or copper lines could be seen, the iron spectrum being exactly the same whether the spark gap was used or not, thus showing that the light from the spark can not pass through the explosion vapors.

This experiment proves that 4 centimeters of iron vapor as here used is perfectly opaque, but it does not show how far the light from the spark was able to penetrate the vapor before it was absorbed, in other words, the absorption coefficient has been shown to be fairly large but has not been measured. Other observations make it probable that a layer about 2 mm. thick is very nearly opaque—but further experimental work is required, and will be undertaken as soon as possible.

[1] *Proc. Nat. Acad. Sci.*, **6** 1920, pp. 42–43; *Astroph. J.*, **51** 1920, pp. 37–48.

[2] The thermo-couple was kindly placed at my disposal by Mr. Edison Pettit.

[3] *Astroph. J.*, **51** 1920, pp. 44–46.

Acknowledgment. Since my Note on the Definition of a Linear Functional was published in the February number, I have learned that the principal theorem stated on page 27 has been proved by Fréchet by means of Fourier series, and that it is also a special case of a theorem by F. Riesz. See *Trans. Amer. Math. Soc.*, **8**, p. 439, and *Math. Ann.*, **69**, p. 475. CHARLES A. FISCHER

Erratum. Page 145, line 20. For "the majority of the stars of" read *about 20% of the stars of the.*